Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 311 (2005) 162–181

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Solvability of nonlinear variational–hemivariational inequalities

Michael E. Filippakis [*,1], Nikolaos S. Papageorgiou

Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece

Received 19 July 2004
Available online 8 April 2005
Submitted by R.E. Showalter

Abstract

In this paper we study nonlinear elliptic differential equations driven by the p-Laplacian with unilateral constraints produced by the combined effects of a monotone term and of a nonmonotone term (variational–hemivariational inequality). Our approach is variational and uses the subdifferential theory of nonsmooth functions and the theory of accretive and monotone operators. Also using these ideas and a special choice of the monotone term, we prove the existence of a strictly positive smooth solution for a class of nonlinear equations with nonsmooth potential (hemivariational inequality).

Keywords: Generalized subdifferential; Convex subdifferential; p-Laplacian; Principal eigenvalue; m-accretive operator; Maximal monotone operator; Critical point

1. Introduction

In this paper we prove an existence result for variational–hemivariational inequalities driven by the p-Laplacian. Then using the argument of the existence theorem and with a particular choice of the monotone (convex) component of the problem, we prove the

* Corresponding author.
E-mail addresses: mfilip@math.ntua.gr (M.E. Filippakis), npapg@math.ntua.gr (N.S. Papageorgiou).
1 Supported by a grant of the National Scholarship Foundation of Greece (I.K.Y.).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.02.029

existence of positive smooth solutions for a class of hemivariational inequalities involving the p-Laplacian differential operator.

So let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. The problem under consideration is the following:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z, x(z)) - \partial G(x(z)) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0, \quad 2 \leqslant p < \infty. \end{cases} \tag{1.1}$$

Here $j(z, x)$ is a measurable function which is locally Lipschitz in the x-variable and $\partial j(z, x)$ denotes the generalized subdifferential of the locally Lipschitz function $x \to j(z, x)$ (see Section 2). Also $G : X \to \bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is proper, convex, lower semicontinuous and $\partial G(x)$ stands for the subdifferential in the sense of convex analysis of the convex function $x \to G(x)$. So in problem (1.1) we have the combined effects of the unilateral constraints imposed by a monotone (convex) term and by a nonmonotone (nonconvex) term. The presence of the $\partial G(x)$-term (the monotone term), classifies the problem as a variational inequality, while the presence of the $\partial j(z, x)$-term (the nonmonotone term) makes the problem a hemivariational inequality. This explains the nomenclature "variational–hemivariational inequality."

Hemivariational inequalities (i.e., $G \equiv 0$), have been studied recently by many authors, primarily in the context of semilinear problems (i.e., $p = 2$) and already there is a substantial literature on the subject. For a detailed bibliography, we refer to Gasinski–Papageorgiou [5]. Hemivariational inequalities (as the generalization of variational inequalities, see Showalter [14]), turned out to be a very useful model in describing many problems in mechanics and engineering involving nonconvex and nonsmooth energy functionals. For various applications, we refer to the book of Naniewicz–Panagiotopoulos [13].

In contrast the study of variational–hemivariational inequalities is lagging behind. There are only the works of Goeleven–Motreanu [6] (semilinear problems with G being an indicator function) and Kyritsi–Papageorgiou [8], Marano–Motreanu [12] and Filippakis–Papageorgiou [4] (problems involving the p-Laplacian and with G being an indicator function).

Our approach is variational and combines notions and techniques from nonsmooth analysis and from nonlinear analysis. In the next section, for the convenience of the reader, we review the basic definitions and results from these areas, which we will be using in our analysis. Our main references are the books of Denkowski–Migorski–Papageorgiou [2,3] and of Showalter [14].

2. Mathematical background

Let X be a Banach space. By X^* we denote its topological dual and by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz,* if for every $x \in X$ we can find U a neighborhood of x and a constant $k_U > 0$ such that

$$\left|\varphi(y) - \varphi(v)\right| \leqslant k_U \|y - v\| \quad \text{for all } y, v \in U.$$

Recall that if $\psi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper (i.e., not identically $+\infty$), convex and lower semicontinuous function, then ψ is locally Lipschitz in the interior of its effective