Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 311 (2005) 182–190

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

A singular measure on the Cantor group ☆

Qi-Yan Shi

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China

Received 8 October 2004

Available online 30 March 2005

Submitted by J. Diestel

Abstract

Let $\Omega = \{-1, 1\}^{\mathbf{N}}$ and $\{\omega_j\}$ be independent random variables taking values in $\{-1, 1\}$ with equal probability. Endowed with the product topology and under the operation of pointwise product, Ω is a compact Abelian group, the so-called Cantor group. Let a, b, c be real numbers with $1 + a + b + c > 0$, $1 + a - b - c > 0$, $1 - a + b - c > 0$ and $1 - a - b + c > 0$. Finite products on Ω,

$$P_n = \prod_{j=1}^{n} (1 + a\omega_j + b\omega_{j+1} + c\omega_j\omega_{j+1}),$$

are studied. We show that the weak limit of $\left\{\frac{P_n\,d\omega}{\int_\Omega P_n\,d\omega}\right\}$ exists in the topology of $M(\Omega)$, where $M(\Omega)$ is the convolution algebra of all Radon measure on Ω, thus defined a probability measure on Ω. We also prove that the measure is continuous and singular with respect to the normalized Haar measure on Ω.

Keywords: Cantor group; Weak topology; Continuity; Singularity

☆ This work is supported by the National Science Foundation of China (grant No. 10471150) and Base Research Foundation of Tsinghua University (grant No. JC2003042).

E-mail address: qshi@math.tsinghua.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.02.031

1. Introduction

Throughout this paper, let

$$\Omega = \prod_{j=1}^{\infty} \Omega_j = \{-1, 1\}^{\mathbf{N}}$$

be the Cartesian product with all factors equal to $\Omega_j = \{-1, 1\}$ ($\forall j \geqslant 1$), and write its elements

$$\varepsilon = (\varepsilon_n)_{n\in\mathbf{N}} \quad \text{or} \quad \varepsilon = \varepsilon_1\varepsilon_2\ldots.$$

Ω is well known as an Abelian group under the operation of pointwise product. With the discrete topology on each factor, the product topology on Ω make it a compact Abelian group, the so-called Cantor group [3]. This topology can also be induced by a metric: the distance between two elements $\varepsilon = (\varepsilon_n)_{n\in\mathbf{N}}$, $\delta = (\delta_n)_{n\in\mathbf{N}}$ in Ω is defined by

$$d(\varepsilon, \delta) = 2^{-\inf\{n\geqslant 0:\ \varepsilon_{n+1}\neq\delta_{n+1}\}}.$$

The dual group Γ of Ω consists of all characters, i.e. continuous group homomorphisms, from Ω into the multiplicative group of complex numbers of modulus 1. Denote the projection $\omega_n : \Omega \to \{-1, 1\}$ by $\omega_n(\varepsilon) = \varepsilon_n$. Characters are provided by the projection functions. Precisely, let $\mathcal{R} = \{\omega_n\colon n \in \mathbf{N}\} \subset \Gamma$, each nontrivial element of Γ can be uniquely written as $\omega_{j_1}\omega_{j_2}\ldots\omega_{j_k}$, where $1 \leqslant j_1 < j_2 < \cdots < j_k < \infty$. Note that for the normalized Haar measure m on Ω which is the product measure of the Haar measures on Ω_j for all $j \in \mathbf{N}$, $\{\omega_j\}$ may be viewed as independent random variables taking values in $\{-1, 1\}$ with equal probability. By abuse of language, we will write dm as $d\omega$, and the Haar measure on $\Omega_j = \{-1, 1\}$ by $d\omega_j$ in the sequel.

Let $M(\Omega)$ be the convolution algebra of all Radon measures on Ω. As usual, we define the Fourier transform of $\mu \in M(\Omega)$ by

$$\hat{\mu}(\gamma) = \int_{\Omega} \gamma\, d\mu, \quad \text{for all } \gamma \in \Gamma.$$

The following result due to Lévy will be needed in the next section.

Theorem A. *Let G be a nondiscrete metrizable compact Abelian group with discrete dual group Γ, $\{\mu_n\}$ be a sequence of probability measures on G. If $\hat{\mu}_n$ converges everywhere in Γ and defines the limit function f. Then μ_n converges weakly to a probability measure μ on G, and $f = \hat{\mu}$.*

If we recall the definition of weak convergence and the fact that the finite linear combinations of members of Γ are dense in $C(G)$, the space of continuous functions on G, the proof of Lévy theorem is quite immediate.

In Ref. [1] we studied a measure on Ω defined by

$$\frac{\prod_{j=1}^{\infty}(1 + a\omega_j + b\omega_{j+1})\, d\omega}{\int_{\Omega}\prod_{j=1}^{\infty}(1 + a\omega_j + b\omega_{j+1})\, d\omega},$$