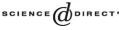


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## A singular measure on the Cantor group $\stackrel{\text{\tiny{thetermat}}}{\longrightarrow}$

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## Abstract

Let  $\Omega = \{-1, 1\}^{\mathbf{N}}$  and  $\{\omega_j\}$  be independent random variables taking values in  $\{-1, 1\}$  with equal probability. Endowed with the product topology and under the operation of pointwise product,  $\Omega$  is a compact Abelian group, the so-called Cantor group. Let a, b, c be real numbers with 1 + a + b + c > 0, 1 + a - b - c > 0, 1 - a + b - c > 0 and 1 - a - b + c > 0. Finite products on  $\Omega$ ,

$$P_n = \prod_{j=1}^n (1 + a\omega_j + b\omega_{j+1} + c\omega_j \omega_{j+1}),$$

are studied. We show that the weak limit of  $\left\{\frac{P_n d\omega}{\int_{\Omega} P_n d\omega}\right\}$  exists in the topology of  $M(\Omega)$ , where  $M(\Omega)$  is the convolution algebra of all Radon measure on  $\Omega$ , thus defined a probability measure on  $\Omega$ . We also prove that the measure is continuous and singular with respect to the normalized Haar measure on  $\Omega$ .

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## 1. Introduction

Throughout this paper, let

$$\Omega = \prod_{j=1}^{\infty} \Omega_j = \{-1, 1\}^{\mathbf{N}}$$

be the Cartesian product with all factors equal to  $\Omega_j = \{-1, 1\} \ (\forall j \ge 1)$ , and write its elements

$$\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$$
 or  $\varepsilon = \varepsilon_1 \varepsilon_2 \dots$ 

 $\Omega$  is well known as an Abelian group under the operation of pointwise product. With the discrete topology on each factor, the product topology on  $\Omega$  make it a compact Abelian group, the so-called Cantor group [3]. This topology can also be induced by a metric: the distance between two elements  $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ ,  $\delta = (\delta_n)_{n \in \mathbb{N}}$  in  $\Omega$  is defined by

$$d(\varepsilon, \delta) = 2^{-\inf\{n \ge 0: \varepsilon_{n+1} \ne \delta_{n+1}\}}.$$

The dual group  $\Gamma$  of  $\Omega$  consists of all characters, i.e. continuous group homomorphisms, from  $\Omega$  into the multiplicative group of complex numbers of modulus 1. Denote the projection  $\omega_n : \Omega \to \{-1, 1\}$  by  $\omega_n(\varepsilon) = \varepsilon_n$ . Characters are provided by the projection functions. Precisely, let  $\mathcal{R} = \{\omega_n : n \in \mathbf{N}\} \subset \Gamma$ , each nontrivial element of  $\Gamma$  can be uniquely written as  $\omega_{j_1}\omega_{j_2}\ldots\omega_{j_k}$ , where  $1 \leq j_1 < j_2 < \cdots < j_k < \infty$ . Note that for the normalized Haar measure *m* on  $\Omega$  which is the product measure of the Haar measures on  $\Omega_j$  for all  $j \in \mathbf{N}$ ,  $\{\omega_j\}$  may be viewed as independent random variables taking values in  $\{-1, 1\}$  with equal probability. By abuse of language, we will write dm as  $d\omega$ , and the Haar measure on  $\Omega_j = \{-1, 1\}$  by  $d\omega_j$  in the sequel.

Let  $M(\Omega)$  be the convolution algebra of all Radon measures on  $\Omega$ . As usual, we define the Fourier transform of  $\mu \in M(\Omega)$  by

$$\hat{\mu}(\gamma) = \int_{\Omega} \gamma \, d\mu$$
, for all  $\gamma \in \Gamma$ .

The following result due to Lévy will be needed in the next section.

**Theorem A.** Let G be a nondiscrete metrizable compact Abelian group with discrete dual group  $\Gamma$ ,  $\{\mu_n\}$  be a sequence of probability measures on G. If  $\hat{\mu}_n$  converges everywhere in  $\Gamma$  and defines the limit function f. Then  $\mu_n$  converges weakly to a probability measure  $\mu$  on G, and  $f = \hat{\mu}$ .

If we recall the definition of weak convergence and the fact that the finite linear combinations of members of  $\Gamma$  are dense in C(G), the space of continuous functions on G, the proof of Lévy theorem is quite immediate.

In Ref. [1] we studied a measure on  $\Omega$  defined by

$$\frac{\prod_{j=1}^{\infty} (1 + a\omega_j + b\omega_{j+1}) d\omega}{\int_{\Omega} \prod_{j=1}^{\infty} (1 + a\omega_j + b\omega_{j+1}) d\omega}$$

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