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Blow-up rate estimates for a doubly coupled reaction–diffusion system ☆

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Abstract

This paper deals with a reaction–diffusion system with coupled nonlinear inner sources and a nonlinear boundary flux. Blow-up rates are determined for four different blow-up situations. The so-called characteristic algebraic system is introduced to get a very simple and clear description for the desired blow-up rate estimates. It is pointed out that one cannot directly use super and sub-solutions to establish blow-up rate estimates, since they do not share the same blow-up time in general.

Keywords: Reaction–diffusion system; Characteristic algebraic system; Blow-up rate

1. Introduction

In this paper, we consider the following reaction–diffusion equations coupled via both nonlinear sources and a nonlinear boundary flux:

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$$\begin{cases} u_t = u_{xx} + u^{l_{11}} v^{l_{12}}, \quad v_t = v_{xx} + u^{l_{21}} v^{l_{22}}, \\ \quad (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) = (u^{p_{11}} v^{p_{12}})(1,t), \quad v_x(1,t) = (u^{p_{21}} v^{p_{22}})(1,t), & t \in (0,T), \\ u_x(0,t) = 0, \quad v_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $l_{ij}, p_{ij} \geqslant 0$ $(i,j=1,2)$; $u_0(x)$ and $v_0(x)$ are smooth functions satisfying the compatible conditions. We deal with completely coupled cases only: it is required that at least one of $l_{12}l_{21}$, $l_{12}p_{21}$, $p_{12}l_{21}$ and $p_{12}p_{21}$ is positive.

Global existence and nonexistence of positive solutions of (1.1) can be found in [26,33]. It was proved that the solutions of (1.1) blow up in finite time if and only if at least one of the following conditions holds: $l_{11} > 1$; $l_{22} > 1$; $p_{11} > 1$; $p_{22} > 1$; $p_{12}p_{21} > (1-p_{11})(1-p_{22})$; $l_{12}l_{21} > (1-l_{11})(1-l_{22})$; $l_{12}p_{21} > (1-l_{11})(1-p_{22})$; $p_{12}l_{21} > (1-p_{11})(1-l_{22})$.

More special cases of (1.1) were studied by many authors. The heat equations coupled via a nonlinear boundary flux

$$\begin{cases} u_t = \Delta u, \quad v_t = \Delta v & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}}, \quad \frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}} & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.2}$$

were studied by Rossi [25], Pederson and Lin [21]. The blow-up rates of radial solutions of (1.2) with large initial data and $\Omega = B_R$ were known as follows:

$$c \leqslant \sup_{B_R} u(\cdot,t)(T-t)^{\tilde{\alpha}_1} \leqslant C, \qquad c \leqslant \sup_{B_R} v(\cdot,t)(T-t)^{\tilde{\beta}_1} \leqslant C,$$

$$\tilde{\alpha}_1 = \frac{p_{12}+1-p_{22}}{2[p_{12}p_{21}-(1-p_{11})(1-p_{22})]}, \qquad \tilde{\beta}_1 = \frac{p_{21}+1-p_{11}}{2[p_{12}p_{21}-(1-p_{11})(1-p_{22})]}.$$

The case for more general domains was considered by Chen [1], and the case with $p_{ii} = 0$ $(i=1,2)$ of (1.2) was studied by Deng [3]. Scalar cases of (1.2) were well studied in [4,11–14].

The blow-up rates of radial solutions to the homogeneous Dirichlet problem of coupled reaction–diffusion equations

$$u_t = \Delta u + u^{l_{11}} v^{l_{12}}, \qquad v_t = \Delta v + u^{l_{21}} v^{l_{22}} \quad \text{in } B_R \times (0,T) \tag{1.3}$$

were obtained by Zheng [35] and Wang [28] as

$$c \leqslant \sup_{B_R} u(\cdot,t)(T-t)^{\tilde{\alpha}_2} \leqslant C, \qquad c \leqslant \sup_{B_R} v(\cdot,t)(T-t)^{\tilde{\beta}_2} \leqslant C,$$

$$\tilde{\alpha}_2 = \frac{l_{12}+1-l_{22}}{l_{12}l_{21}-(1-l_{11})(1-l_{22})}, \qquad \tilde{\beta}_2 = \frac{l_{21}+1-l_{11}}{l_{12}l_{21}-(1-l_{11})(1-l_{22})}.$$

The studies for scalar cases of (1.3) can be found in [6,8–10]. A special case of (1.3) with $l_{ii} = 0$ $(i=1,2)$ was discussed by Wang [30].

Fu and Guo [7], Wang [31] considered blow-up rates and sets for system (1.1) with $l_{ii} = p_{ii} = 0$ $(i=1,2)$. The other cases for (1.1) with $l_{21} = p_{12} = l_{ii} = p_{ii} = 0$ $(i=1,2)$ and $l_{21} = l_{ii} = p_{ii} = 0$ $(i=1,2)$ were studied by Wang also [29,32].