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The asymptotic behaviour of the unique solution for the singular Lane–Emden–Fowler equation ☆

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Abstract

By Karamata regular variation theory and constructing comparison functions, we show the exact asymptotic behaviour of the unique classical solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ near the boundary to a singular Dirichlet problem $-\Delta u = k(x)g(u)$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$; $g \in C^1((0,\infty),(0,\infty))$, $\lim_{t\to 0^+} \frac{g(\xi t)}{g(t)} = \xi^{-\gamma}$, for each $\xi > 0$, for some $\gamma > 0$; and $k \in C^\alpha_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$, is nonnegative on Ω, which is also singular near the boundary.

Keywords: Semilinear elliptic equations; Dirichlet problem; Singularity; Karamata regular variation theory; Unique solution; Existence; Exact asymptotic behaviour

1. Introduction and the main results

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geqslant 1$). Consider the following singular Dirichlet problem for the Lane–Emden–Fowler equation:

$$-\Delta u = k(x)g(u), \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{1.1}$$

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where g and k satisfy

(g_1) $g \in C^1((0,\infty),(0,\infty))$, $g'(s) \leqslant 0$ for all $s > 0$, $\lim_{s\to 0^+} g(s) = +\infty$;
(k_1) $k \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$, is nonnegative and nontrivial on Ω.

The problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically materials (see [10,13,17,25,29]).

The main feature of this paper is the presence of the singular term $g(u)$ which is regular varying at zero of index $-\gamma$ with $\gamma > 0$, and the weight $k(x)$ which is also singular near the boundary, the both of them include a large class of singular functions.

The problem was discussed and extended to the more general problems in number of works; see, for instance, [2,9–17,19–21,23,28–33]. For $k \equiv 1$ on Ω, and g satisfying (g_1), Crandall et al. [10, Theorems 2.2 and 2.7] showed that problem (1.1) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, there exist positive constants C_1 and C_2 such that

(I) $C_1 p(d(x)) \leqslant u(x) \leqslant C_2 p(d(x))$ near $\partial\Omega$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$,

where $p \in C[0,a] \cap C^2(0,a]$ is the local solution to the problem

$$-p''(s) = g\big(p(s)\big), \quad p(s) > 0,\ 0 < s < a,\ p(0) = 0. \tag{1.2}$$

In particular, for $g(u) = u^{-\gamma}$, $\gamma > 1$, u satisfies

(I_1) $C_1[d(x)]^{2/(1+\gamma)} \leqslant u(x) \leqslant C_2[d(x)]^{2/(1+\gamma)}$ near $\partial\Omega$.

Lazer and McKenna [21], by construction the global subsolution and supersolution, showed that (I_1) continues to hold on $\bar{\Omega}$. Then $u \in H^1_0(\Omega)$ if and only if $\gamma < 3$. This is a basic character to problem (1.1). Moreover, in the Remarks and generalizations, there is the following additional information:

(I_2) if, instead of $k(x) \equiv 1$ on Ω, assume $0 < c_1 \leqslant k(x)\varphi_1^{\sigma}(x) \leqslant c_2$ for all $x \in \bar{\Omega}$, where c_1 and c_2 are constants, $0 < \sigma < 2$, and φ_1 is the eigenfunction corresponding to the first eigenvalue of problem $-\Delta u = \lambda u$ in Ω, and $u|_{\partial\Omega} = 0$, then for $\gamma > 1$, there exist positive constants C_1 and C_2 (C_1 small and C_2 large) such that u satisfying

$$C_1\big[\varphi_1(x)\big]^{2/(1+\gamma)} \leqslant u(x) \leqslant C_2\big[\varphi_1(x)\big]^{(2-\sigma)/(1+\gamma)}, \quad \forall x \in \bar{\Omega}.$$

Most recently, in [32], we showed the existence and global optimal estimate of unique solution to problem (1.1) under $\int_1^{\infty} g(s)\,ds < \infty$. Moreover, Ghergu and Rădulescu [14] showed that if g satisfies (g_1) and

(g_2) there exist positive constants C_0, η_0 and $\gamma \in (0,1)$ such that $g(s) \leqslant C_0 s^{-\gamma}$, $\forall s \in (0,\eta_0)$;
(g_3) there exist $\theta > 0$ and $t_0 \geqslant 1$ such that $g(\xi t) \geqslant \xi^{-\theta} g(t)$ for all $\xi \in (0,1)$ and $0 < t \leqslant t_0\xi$;