Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 312 (2005) 357–373

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Existence of solutions to a class of nonlinear second order two-point boundary value problems

Fuyi Li [a,*], Zhanping Liang [a,b], Qi Zhang [a]

[a] *Department of Mathematics, Shanxi University, Taiyuan 030006, People's Republic of China*
[b] *Department of Mathematics, Xinzhou Teachers' University, Xinzhou 034000, Shanxi, People's Republic of China*

Received 9 January 2005

Available online 13 April 2005

Submitted by J. Henderson

Abstract

In this paper, the existence and multiplicity results of solutions are obtained for the second order two-point boundary value problem $-u''(t) = f(t, u(t))$ for all $t \in [0, 1]$ subject to $u(0) = u'(1) = 0$, where f is continuous. The monotone operator theory and critical point theory are employed to discuss this problem, respectively. In argument, quadratic root operator and its properties play an important role.

Keywords: Two-point boundary value problem; Strongly monotone operator principle; Critical point theory; P.S. condition; Mountain pass lemma

* Corresponding author.
E-mail address: fyli@sxu.edu.cn (F. Li).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.03.043

1. Introduction

In this paper, we consider the existence and multiplicity results of the solutions to the following second order two-point boundary value problem (BVP):

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^1 \to \mathbb{R}^1$ is continuous.

Owing to the importance of second order differential equations in physics, the existence and multiplicity of the solutions to the following problem

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u(1) = 0 \end{cases}$$

has been studied by many authors, see [1,3–10]. They all obtained the existence results of positive solutions under that f is either superlinear or sublinear in u by employing the cone expansion or compression fixed point theorem. Meanwhile, great importance has been attached to BVP (1.1). But in our knowledge, few papers have discussed the existence results of solutions, especially, infinitely many solutions for BVP (1.1). In this paper, by using the strongly monotone operator principle and the critical point theory, respectively, to discuss BVP (1.1), we establish some conditions for f which are able to guarantee that this problem has a unique solution, at least one nonzero solution, and infinitely many solutions. In argument, $K^{1/2}$, the quadratic root operator of a positive linear compact operator K, and its properties play an important role.

2. Preliminary

In this section, we give some lemmas that are important to our discussion. Let $C[0,1]$ denote the usual real Banach space with the norm $\|u\|_C = \max_{t\in[0,1]} |u(t)|$ for all $u \in C[0,1]$, $L^2[0,1]$ denote the usual real reflexive Banach space with the norm $\|u\| = (\int_0^1 |u(t)|^2\,dt)^{1/2}$ for all $u \in L^2[0,1]$ and the real Hilbert space with the inner product $(u, v) = \int_0^1 u(t)v(t)\,dt$ for all $u, v \in L^2[0,1]$.

It is well known that any solution of BVP (1.1) in $C^2[0,1]$ is equivalent to a solution of the following integral equation in $C[0,1]$,

$$u(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\,ds, \quad t \in [0,1], \tag{2.1}$$

where $G : [0,1] \times [0,1] \to [0,1]$ is Green's function for $-u''(t) = 0$ for all $t \in [0,1]$ subject to $u(0) = u'(1) = 0$, i.e.,

$$G(t,s) = \min\{t,s\} = \begin{cases} t, & 0 \leqslant t \leqslant s \leqslant 1, \\ s, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Define operators $K, \mathbf{f} : C[0,1] \to C[0,1]$ respectively by