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J. Math. Anal. Appl. 311 (2005) 479–488
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Nonclassical symmetries of a class of nonlinear partial differential equations with arbitrary order and compatibility

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Received 31 October 2004
Available online 31 March 2005
Submitted by N.H. Ibragimov

Abstract

In this paper, we show that for a class of nonlinear partial differential equations with arbitrary order the determining equations for the nonclassical reduction can be obtained by requiring the compatibility between the original equation and the invariant surface condition. The nonlinear wave equation and the Boussinesq equation all serve as examples illustrating this fact.

Keywords: Nonclassical symmetries; Compatibility; Determining equations; Nonlinear wave equation; Boussinesq equation

1. Introduction

The nonclassical method of reduction was devised originally by Bluman and Cole, in 1969, to find new exact solutions of the heat equation in Ref. [5]. Let us first reformulate the "nonclassical method" of reduction (in Ref. [5]) in terms of vector fields and their prolongations. The nonclassical method could be used for an arbitrary system of differential

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.03.004

equations, but for the purposes of this paper, we restrict ourselves to one nth-order PDE of $(1+1)$-dimension as follows:

$$\Delta(x,t,u,u_t,u_x,u_{tt},u_{tx},\ldots)=0. \tag{1}$$

Suppose the form of Eq. (1) is invariant under a group action on (x,t,u) space given by its infinitesimals

$$\begin{aligned} x^* &= x + X(x,t,u)\epsilon + O(\epsilon^2),\\ t^* &= t + T(x,t,u)\epsilon + O(\epsilon^2),\\ u^* &= u + U(x,t,u)\epsilon + O(\epsilon^2). \end{aligned} \tag{2}$$

The invariance requirement is

$$\Gamma^{(n)}\Delta|_{\Delta=0}=0, \tag{3}$$

where $\Gamma^{(n)}$ is the nth extension of the infinitesimal generator

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}. \tag{4}$$

Solving Eq. (3) leads to the infinitesimals X, T and U for the classical Lie point symmetry. The nonclassical method seeks the invariance of the original equation (1) augmented with the invariant surface condition

$$\Delta_0 = Xu_x + Tu_t - U = 0. \tag{5}$$

The nonclassical symmetries are determined by the governing equation

$$\Gamma^{(n)}\Delta|_{\Delta=0,\,\Delta_0=0}=0. \tag{6}$$

Solving this governing equation leads to a set of the determining equations for the infinitesimals X, T and U. When the determining equations are solved, that gives rise to the nonclassical symmetries of Eq. (1). Solutions of Eq. (5) leads to a solution Ansatz, which, when substituted into Eq. (1) gives a reduction of the original equation. When the reduced equation is solved, then we obtain the invariant solutions under group (2) of the original equation (1). The solutions obtained by the nonclassical method and the classical method are different. So the method of nonclassical reduction has been used to find new exact solutions of nonlinear partial differential equations of physical and mathematical interest. Moreover, this method leads to the extensions of the symmetry method in Refs. [1,2], for example, conditional symmetries and partial symmetries. These symmetries all involve differential constraints. Recently in Ref. [6] Broadbridge and Arrigo have shown that all solutions of standard symmetric linear partial differential equations have classical Lie symmetry. In addition to the nonclassical method, the determining equations are usually highly nonlinear unlike the determining equations for the classical method which are linear (see Ref. [4]). The properties and relationships of the nonclassical reduction, the determining equations of it and the invariant surface condition is worth while to study. We note that Arrigo and Beckham in Ref. [3] show that the determining equations for the nonclassical method can be derived as a consequence of the compatibility for the evolutionary partial differential equations. In this paper, we show that for a class of the nonlinear PDE with arbitrary order instead of the nonlinear evolution equations, the determining equations for