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Value sharing and differential equations

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Abstract

Suppose that f is a nonconstant entire function and $L[f]$ a linear differential polynomial in f with constant coefficients. In this paper, by considering the existence of the solutions of some differential equations, we find all the forms of entire functions f in most cases when f and $L[f]$ share two values counting multiplicities jointly. This result generalize some known results due to Rubel–Yang and Li–Yang.

Keywords: Entire function; Differential equation; Value sharing

1. Introduction and results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and c a finite complex number. If $f(z)-c$ and $g(z)-c$ have the same zeros counting multiplicities (ignoring multiplicities), then we say that $f(z)$ and $g(z)$ share the value c CM (IM). We say $f(z)$ and $g(z)$ share ∞ CM (IM) if $1/f$ and $1/g$ share 0 CM (IM). Let S be a subset of distinct elements in $\mathbb{C}$. Define

$$E_f(S)=\bigcup_{a\in S}\bigl\{z \mid f(z)-a=0,\ \text{counting multiplicities}\bigr\},$$

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$$\tilde{E}_f(S) = \bigcup_{a\in S}\{z \mid f(z) - a = 0, \text{ ignoring multiplicities}\}.$$

We say that $f(z)$ and $g(z)$ share the set S CM (IM) provided that $E_f(S) = E_g(S)$ ($\tilde{E}_f(S) = \tilde{E}_g(S)$). It was shown [11] that if an entire function f and its derivative f' share two finite values a, b CM, then $f \equiv f'$. Mues–Steinmetz [8], and G. Gundersen [4] independently, improved this result. They proved that the conclusion remains to be valid if f share two values IM with f' only. This result has been generalized to the case that f share two values with a linear differential polynomial in f by several mathematicians, see, e.g., [1] and [9,10]. Li–Yang [7] considered the case that f share two values CM jointly with its derivative, and proved that if f and f' share the set $\{a_1, a_2\}$ CM, then f assume one of the following cases: (i) $f \equiv f'$; (ii) $f + f' \equiv a_1 + a_2$; (iii) $f \equiv c_1e^{cz} + c_2e^{-cz}$, with $a_1 + a_2 = 0$, where c, c_1 and c_2 are nonzero constants which satisfy $c^2 \neq 1$ and $c_1c_2 = \frac{1}{4}a_1^2(1 - c^{-2})$.

It is natural to ask what will be happen when f share two values jointly with its linear differential polynomial. In the present paper, we shall prove the following result.

Theorem 1. *Suppose that f is a nonconstant entire function and*

$$g = L_n[f] = b_{-1} + \sum_{i=0}^{n} b_i f^{(i)}, \tag{1}$$

where b_i $(i = -1, 0, 1, \ldots, n)$ are constants and $b_n \neq 0$. Let a_1 and a_2 be two distinct numbers in $\mathbb{C}$. If f and g share the set $\{a_1, a_2\}$ CM, then one of the following cases holds:

(i) $f = g$;
(ii) $f + g = a_1 + a_2$;
(iii) $f = \frac{a_1+a_2}{2} + \frac{h_1}{2} + \frac{h_2}{2}$, $g = \frac{a_1+a_2}{2} - \frac{1}{2}e^{\gamma}h_1 + \frac{1}{2}e^{\gamma}h_2$, *where γ is an entire function such that $T(r, e^{\gamma}) = S(r, f)$ and h_1, h_2 are two entire functions satisfying $h_1h_2 = (\frac{a_1-a_2}{2})^2(1 - e^{-2\gamma})$, and the following two differential equations*:

$$\sum_{k=0}^{n} b_k h_1^{(k)} = -e^{\gamma}h_1, \qquad \sum_{k=0}^{n} b_k h_2^{(k)} = e^{\gamma}h_2. \tag{2}$$

Moreover, $2b_{-1} + (a_1 + a_2)(b_0 - 1) = 0$.

By considering the existence of the solutions of the two equations in (2), we also give the explicit forms of h_1 and h_2 for some special cases.

Theorem 2. *Suppose that f is a nonconstant entire function and $g = L_n[f]$ is the differential polynomial defined in* (1), *where $n = 1$ or n is an even number. Let a_1 and a_2 be two distinct numbers in $\mathbb{C}$. If f and g share the set $\{a_1, a_2\}$ CM, then one of the following cases holds*:

(i) $f = g$;
(ii) $f + g = a_1 + a_2$;