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Existence results for a degenerated nonlinear elliptic partial differential equation

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Abstract

The aim of this paper is to establish the existence of weak solutions to a steady state two-dimensional irrotational compressible flow around a thin profile. This flow is described by the small disturbance equations. If the speed of sound exceeds the fluid one, the governing equations remain elliptic. But when the fluid speed is beyond the sound one, the flow becomes locally hyperbolic and shock waves arise. For a modified elliptic model, using convexity arguments, we prove the existence of a solution which is the solution to the first model when the flow remains subsonic.

Keywords: Degenerated elliptic equation; Small disturbance equations; Convex analysis; Dual problem

1. Introduction

The transonic flows of perfect compressible fluids pose fundamental problems. Indeed, only partial results concerning the existence and uniqueness of solutions are proved. In [1,11], the authors give a functional method to solve the equations governing the speed field of subsonic flows. They used an iterative algorithm and proved in [1] that if the flow

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is strictly subsonic, the problem has a unique solution. The proof is based on a fixed point theorem. But if the flow is transonic, this theorem does not work any more. A different technique is given in [4] by Gittel. The author uses the concept of compactness by compensation (see [9]) and looks for a solution satisfying certain constraint (see [10] for this notion).

The type of flow is variable, i.e. whether the flow is subsonic, sonic or supersonic, the corresponding equation is elliptic, parabolic or hyperbolic, respectively. The difficulties of the problem come from the nonlinearity of the equations and the change of the flow type. The position of the type change (sonic line) is unknown a priori. The presence of jump lines (shock waves), whose positions are also unknown, increases the precedent difficulties. Many problems are represented by this model, in particular, airflow around a thin aerodynamic profile or gas flows in a channel.

Since forty years, many numerical and mathematical researches have been done in this field. From theoretical point of view, there are not complete results to ensure the existence or uniqueness of solutions. Currently, only methods based on conjectures are performed [4,7,8]. But from numerical point of view, many simulations are realized, the results obtained are in accordance with the experimentation, in particular the existence of shock waves (see, for example, [2,5,6]).

In the present paper we consider a two-dimensional irrotational compressible flow around a thin profile placed in an infinite atmosphere. This flow is assumed to be uniform at infinity and it will be considered as a perturbation of this flow at infinity. After an asymptotical analysis, one can restrict the study domain, by truncation, to a bounded domain $\Omega =]-R_x, R_x[\times]0, R_y[$. The lengths R_x and R_y are chosen large enough to display the boundary conditions at infinity on the truncated boundary. If one denotes by (u, v) the vector valued function which describes the velocity of the flow, the corresponding subsonic small disturbance model is given by

$$\begin{cases} \partial_x g(u) + \partial_y v = 0 & \text{in } \Omega, \\ \partial_x v - \partial_y u = 0 & \text{in } \Omega, \\ v = \sqrt{M_\infty}\, z'(x) & \text{on } \Gamma_p, \\ u = 0 & \text{on } \Gamma_2, \\ v = 0 & \text{on } \Gamma_0 \cup \Gamma_1, \end{cases} \tag{1}$$

where

$$\begin{cases} \Gamma_p = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \{0\}, \\ \Gamma_2 = \{-R_x\} \times]0, R_y[\cup \{R_x\} \times]0, R_y[, \\ \Gamma_0 = \left]-R_x, -\frac{1}{2}\right[\times \{0\} \cup \left]\frac{1}{2}, R_x\right[\times \{0\}, \\ \Gamma_1 =]-R_x, R_x[\times \{R_y\}, \end{cases}$$

and where the function g is defined on $\mathbb{R}$ by

$$g(t) = \begin{cases} \frac{\gamma+1}{2}(u_{\mathrm{cr}} - t)^2 + \frac{\gamma+1}{2} u_{\mathrm{cr}}^2 & \text{if } t \leqslant u_{\mathrm{cr}}, \\ \frac{\gamma+1}{2} u_{\mathrm{cr}}^2 & \text{if } t \geqslant u_{\mathrm{cr}}. \end{cases}$$

$M_\infty \in]0, 1[$ is the Mach number at infinity and $z = z(x)$ describes the equation of the normalized profile. The constants γ and u_{cr} are positive ($\gamma = 1.4$ for the air and u_{cr} represents the critical speed).