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On a nonlinear elliptic eigenvalue problem

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Abstract

Let $N(\lambda)$ be the number of the solutions of the equation: $-\Delta u = \lambda f(u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary. Under suitable conditions on f, we proved that $N(\lambda) \to +\infty$ as $\lambda \to +\infty$.

Keywords: Nonlinear eigenvalue problem; Perturbation from symmetry; Minimax theorem

1. Introduction

In this paper we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$) with smooth boundary, $\lambda > 0$ is a parameter, and $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$$f \in C(\mathbb{R}) \quad \text{and} \quad \lim_{t\to 0} \frac{f(t)}{|t|^{p-2}t} = 1 \quad \text{for some } 2 < p < 2^* := \begin{cases} \frac{2N}{N-2}, & N \geqslant 3, \\ +\infty, & N = 2. \end{cases} \tag{F}$$

Let $N(\lambda)$ be the number of solutions of (P_λ). Our main result is the following theorem.

Theorem 1.1. *If f satisfies condition* (F), *then* $\lim_{\lambda\to+\infty} N(\lambda) = +\infty$.

Remark 1.2. From condition (F), one can see that neither global growth condition nor global symmetric condition is imposed on the nonlinear term f. Thus Theorem 1.1 improves previous results (for example, [1]) on this problem greatly.

Notation. The norm of $L^r(\Omega)$ $(1 < r < \infty)$ and $H_0^1(\Omega)$ is denoted by $|\cdot|_r$ and $\|\cdot\|$ respectively, that is, $|u|_r = (\int_\Omega |u|^r\,dx)^{1/r}$ and $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$.

2. Some lemmas

Let $\tilde{g}(t) = f(t) - |t|^{p-2}t$, then by condition (F) we have

$$\lim_{t\to 0} \frac{\tilde{g}(t)}{|t|^{p-2}t} = 0.$$

So if $a > 0$ small enough, there exists $C_a > 0$ such that

$$\left|\tilde{g}(t)\right| \leqslant C_a |t|^{p-1}, \quad t \in [-2a, 2a], \quad \text{and} \quad C_a \to 0 \quad \text{as } a \to 0. \tag{2.1}$$

Let $a > 0$ be small enough such that

$$0 < C_a < \frac{p-2}{2(p+2)}, \tag{2.2}$$

and (2.1) is satisfied. Let β_a be a C^∞ function satisfying that $\beta_a(t) = 1$ if $|t| \leqslant a$, $\beta_a(t) = 0$ if $|t| \geqslant 2a$, and $0 \leqslant \beta_a(t) \leqslant 1$ for any $t \in \mathbb{R}$. Define

$$g_a(t) = \beta_a(t)\tilde{g}(t), \quad t \in \mathbb{R}$$

and consider the following equation:

$$\begin{cases} -\Delta u = |u|^{p-2}u + \lambda^{(p-1)/(p-2)} g_a\left(\lambda^{-1/(p-2)}u\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$Q_{a,\lambda}$}$$

By the definition of g_a and direct calculation, we have the following lemma.

Lemma 2.1. *If v is a solution of $(Q_{a,\lambda})$ and $|v|_{L^\infty(\Omega)} \leqslant a\lambda^{1/(p-2)}$, then $u(x) = \lambda^{-1/(p-2)}v(x)$, $x \in \Omega$ is a solution of (P_λ).*

Remark 2.2. In the recent paper [5], S.J. Li and Z.L. Liu considered some problems similar to $(Q_{a,\lambda})$. They used some methods coming from non-smooth critical point theory to get