Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 304 (2005) 147–157

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On the manifold of tripotents in JB*-triples

José M. Isidro [a,*,1], László L. Stachó [b,2]

[a] *Facultad de Matemáticas, Universidad de Santiago, 15706 Santiago de Compostela, Spain*
[b] *Bolyai Institute, Aradi Vértanúk tere 1, 6720 Szeged, Hungary*

Received 5 November 2003

Available online 19 November 2004

Submitted by B. Bongiorno

Abstract

The manifold of tripotents in an arbitrary JB*-triple Z is considered, a natural affine connection is defined on it in terms of the Peirce projections of Z, and a precise description of its geodesics is given. Regarding this manifold as a fiber space by Neher's equivalence, the base space is a symmetric Kähler manifold when Z is a classical Cartan factor, and necessary and sufficient conditions are established for connected components of the manifold to admit a Riemann structure.

Keywords: JB*-triples; Cartan factors; Grassmann manifolds; Banach–Lie algebras and groups; Riemann manifolds

1. Introduction

In [9] Hirzebruch proved that the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra is a compact Riemann symmetric space of rank 1, and that any such space arises in this way. Later on, in [14] Nomura estab-

* Corresponding author.
E-mail addresses: jmisidro@zmat.usc.es (J.M. Isidro), stacho@math.u-szeged.hu (L.L. Stachó).
1 Supported by Ministerio de Educación y Cultura of Spain, Research Project BFM2002-01529.
2 Supported by the Bilateral Spanish–Hungarian Project E-50/2002 and Hungarian Research Grant OTKA T34267.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.09.009

lished similar results for the manifold of minimal projections in a topologically simple real Jordan–Hilbert algebra. Recently, Jordan algebras and projections have been replaced by the more general notions of JB*-triples and tripotents, respectively. JB*-triples are precisely those complex Banach spaces whose open unit balls are homogeneous with respect to biholomorphic transformations.

In [1] an affine connection ∇ on $\mathcal{M}$, the manifold of tripotents in a JB*-triple Z, was defined in terms of the natural algebraic triple product structure of Z. Unfortunately, the description of the geodesics of ∇ given in [1, Theorem 2.7] by means of one-parameter groups of automorphisms of Z fails to be true in general since the corresponding second order differential equation is of sophisticated character. Our first goal is to develop a technique, based on exponential integrals, to find explicit formulas for the geodesics of ∇.

It is known that $\mathcal{M}$ is a fibre space with respect to Neher's relation of equivalence of tripotents. As proved by Kaup in [11], the base space $\mathbb{P}$ of that fibration is the manifold of all complemented principal inner ideals of Z, which is a closed complex submanifold of the Grassmannian $\mathbb{G} = \mathbb{G}(Z)$. The connected components of $\mathbb{P}$, which are orbits of Γ (the structure group of Z), are symmetric complex Banach manifolds on which Γ acts as a group of isometries, see [11]. We show that ∇ induces on these orbits a Γ-invariant torsion-free affine connection (also denoted by ∇) and compute its geodesics which turn out to be orbits of one-parameter subgroups of Γ.

All tripotents in the same equivalence class (in Neher's sense) have the same rank r ($0 \leqslant r \leqslant \infty$), that is constant over each connected component M of $\mathbb{P}$. It is reasonable to ask which of these connected components admit a Riemann structure. For Z a classical Cartan factor, we solve that problem with the aid of the concepts of *operator rank* and *operator corank*, and prove that M admits a Riemann structure if and only if either the operator rank or the operator corank are finite, in which case we prove that ∇ is the Levi-Civita and the Kähler connection of M. Some of these results were already known and due to E. Cartan in the $\mathbb{C}^n$ setting.

2. JB*-triples and tripotents

For a complex Banach space Z, denote by $\mathcal{L}(Z)$ the Banach algebra of all bounded linear operators on Z. A complex Banach space Z with a continuous mapping $(a, b, c) \mapsto \{abc\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following conditions are satisfied for all $a, b, c, d \in Z$, where the operator $a \,\square\, b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\,,\,]$ is the commutator product:

(1) $\{abc\}$ is symmetric complex linear in a, c and conjugate linear in b.
(2) $[a \,\square\, b,\ c \,\square\, d] = \{abc\} \,\square\, d - c \,\square\, \{dab\}$.
(3) $a \,\square\, a$ is hermitian and has spectrum $\geqslant 0$.
(4) $\|\{aaa\}\| = \|a\|^3$.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. An *automorphism* is a bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}$ for $z \in Z$ which occurs if and only if ϕ is a surjective linear isometry of Z.