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Multiple solutions of nonlinear elliptic equations for oscillation problems ☆

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Abstract

We discussed oscillating equations with Neumann boundary value in [Nonlinear Anal. 54 (2003) 431–443] and [J. Math. Anal. Appl. 298 (2004) 14–32] and prove the existence of infinitely many nonconstant solutions. However, it seems difficult to find infinitely many disjoint order intervals for oscillating equations with Dirichlet boundary value. To get rid of this difficulty, in this paper, we build up a mountain pass theorem in half-order intervals and use it to study oscillating problems with Dirichlet boundary value in which we only have the existence of super-solutions (or sub-solutions) and obtain new results on the exactly infinitely many solutions.

Keywords: Mountain pass theorem; Half-order intervals; Oscillating problems

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1. Introduction

In this paper we consider some minimax theorems in order intervals which sharpen some results given in [8], [9] and [1]. As application we study the oscillating nonlinear problems and obtain some existence of multiple solutions for semilinear elliptic boundary value problems.

Let E be a subspace of Hilbert space F and $P_F \subset E$ be a closed convex cone. Let $X \subset E$ be a Banach space which is densely embedded to E. Let $P = X \cap P_F$ and assume that P has nonempty interior $\mathring{P} \neq \emptyset$. We assume any order interval is finitely bounded, Φ is a functional from E to R and satisfies the following assumptions:

(Φ_1) $\Phi \in C^{2-0}(E, R)$ and satisfies the (PS) condition in E. Φ also satisfies so-called deformation property in X which is defined by the following

Definition 1.1. Assume $\Phi \in C^1(X, R)$, $c \in R$. N is a closed neighborhood of $\tilde{K}_c$, where $\tilde{K}_c = \{x \in E \mid \Phi(x) = c,\ \Phi'(x) = 0\}$. If $\forall \varepsilon^* > 0$ and $\forall N$, there exist $\varepsilon \in (0, \varepsilon^*)$ and a continuous map $\eta : [0, 1] \times X \to X$, such that

(i) $\eta(0, \cdot) = \mathrm{id}$;
(ii) $\eta(t, u) = u$, $\forall u \notin \Phi^{-1}[c - \varepsilon^*, c + \varepsilon^*] = \{u \in X\colon c - \varepsilon^* \leqslant \Phi(u) \leqslant c + \varepsilon^*\}$;
(iii) $\Phi(\eta(\cdot, u))$ is decreasing, $\forall u \in X$;
(iv) $\eta(1, \Phi^{c+\varepsilon} \setminus N) \subset \Phi^{c-\varepsilon}$.

(Φ_2) K is a compact mapping from N to N, where N denotes the space F, E and X. The gradient of Φ is of the form $\nabla\Phi = \mathrm{id} - K_E$. K is strongly order preserving from F to X, i.e., $u > v \Rightarrow K(u) \gg K(v)$ for all u,$v \in F$, where $u \gg v \Leftrightarrow u - v \in \mathring{P}$.
(Φ_3) Φ is bounded from below on any interval in X.

We denote the order interval $\{v \in X \mid v_1 \leqslant v \leqslant v_2,\ v_1, v_2 \in F\}$ by $[v_1, v_2]$, i.e., $[v_1, v_2] = [v_1, v_2]_F \cap X$.

In [7], the following mountain pass theorem in order intervals was given.

Proposition 1.2. *Suppose Φ satisfies (Φ_1)–(Φ_3) and $[v_1, v_2]$, $[w_1, w_2]$ are two pairs of strict sub-solutions and super-solutions of $\nabla\Phi = 0$ with $v_2 < w_1$. Then Φ has a mountain pass point $u_0 \in [v_1, w_2] \setminus ([v_1, v_2] \cup [w_1, w_2])$. More precisely, let v_0 be the maximal minimizer of Φ in $[v_1, v_2]$ and w_0 be the minimal minimizer of Φ in $[w_1, w_2]$, then $v_0 \ll u_0 \ll w_0$. Moreover, $C_1(\Phi, u_0)$, the first critical group of Φ at u_0, is nontrivial.*

The theorem gives the location of the mountain pass point in term of the ordered intervals. The assumption of the theorem is that there exists two pairs of strict sub- super-solutions of $\nabla\Phi = 0$. However, for some oscillating nonlinear elliptic boundary value problems we only have the existence of super-solutions (see [4]). In this paper we will establish a mountain pass theorem in half-order intervals and then use it to study the oscillating nonlinear problems.