Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 308 (2005) 290–302

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials

Qiu-Ming Luo [a], H.M. Srivastava [b,*]

[a] *Department of Mathematics, Jiaozuo University, Jiaozuo City, Henan 454003, People's Republic of China*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

Received 12 January 2005

Available online 9 February 2005

Submitted by William F. Ames

Abstract

The main object of this paper is to give analogous definitions of Apostol type (see [T.M. Apostol, On the Lerch Zeta function, Pacific J. Math. 1 (1951) 161–167] and [H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc. 129 (2000) 77–84]) for the so-called Apostol–Bernoulli numbers and polynomials of higher order. We establish their elementary properties, derive several explicit representations for them in terms of the Gaussian hypergeometric function and the Hurwitz (or generalized) Zeta function, and deduce their special cases and applications which are shown here to lead to the corresponding results for the classical Bernoulli numbers and polynomials of higher order.

Keywords: Bernoulli polynomials; Apostol–Bernoulli polynomials; Apostol–Bernoulli polynomials of higher order; Apostol–Euler polynomials; Apostol–Euler polynomials of higher order; Gaussian hypergeometric function; Stirling numbers of the second kind; Hurwitz (or generalized) Zeta function; Hurwitz–Lerch and Lipschitz–Lerch Zeta functions; Lerch's functional equation

* Corresponding author.
E-mail addresses: luomath@126.com (Q.-M. Luo), hmsri@uvvm.uvic.ca, harimsri@math.uvic.ca (H.M. Srivastava).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.020

1. Introduction, definitions and preliminaries

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [8] and [10, p. 61 et seq.]):

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz} = \sum_{n=0}^\infty B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \big(|z|<2\pi;\ 1^\alpha := 1\big) \tag{1}$$

and

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz} = \sum_{n=0}^\infty E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \big(|z|<\pi;\ 1^\alpha := 1\big), \tag{2}$$

so that, obviously,

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0), \tag{3}$$

where

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \big(\mathbb{N} := \{1,2,3,\ldots\}\big).$$

For the classical Bernoulli numbers B_n and the classical Euler numbers E_n, we readily find from (3) that

$$B_n := B_n(0) = B_n^{(1)}(0) \quad \text{and} \quad E_n := E_n(0) = E_n^{(1)}(0) \quad (n \in \mathbb{N}_0). \tag{4}$$

Some interesting analogues of the classical Bernoulli polynomials and numbers were investigated by Apostol [2, Eq. (3.1), p. 165] and (more recently) by Srivastava [9, pp. 83–84]. We begin by recalling here Apostol's definitions as follows.

Definition 1 (Apostol [2]; see also Srivastava [9]). The Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^\infty \mathcal{B}_n(x;\lambda)\frac{z^n}{n!} \quad \big(|z+\log\lambda|<2\pi\big) \tag{5}$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \quad \text{and} \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \tag{6}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol–Bernoulli numbers.

Apostol [2] not only gave elementary properties of the polynomials $\mathcal{B}_n(x;\lambda)$, but also obtained the following recursion formula of the numbers $\mathcal{B}_n(\lambda)$ (see [2, Eq. (3.7), p. 166]):

$$\mathcal{B}_n(\lambda) = n\sum_{k=0}^{n-1}\frac{k!(-\lambda)^k}{(\lambda-1)^{k+1}}S(n-1,k) \quad \big(n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\backslash\{1\}\big), \tag{7}$$