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Symmetry analysis of initial-value problems

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Abstract

Symmetry analysis is a powerful tool that enables the user to construct exact solutions of a given differential equation in a fairly systematic way. For this reason, the Lie point symmetry groups of most well-known differential equations have been catalogued. It is widely believed that the set of symmetries of an initial-value problem (or boundary-value problem) is a subset of the set of symmetries of the differential equation. The current paper demonstrates that this is untrue; indeed, an initial-value problem may have no symmetries in common with the underlying differential equation. The paper also introduces a constructive method for obtaining symmetries of a particular class of initial-value problems.

Keywords: Symmetry analysis; Initial-value problem; Ordinary differential equation

1. Introduction

More than a century ago, the Norwegian mathematician Sophus Lie developed techniques for finding and using the continuous point symmetries of differential equations. Whilst symmetry analysis remains the most widely-applicable method of constructing exact solutions of differential equations, it has not been particularly successful in the treatment of initial- and boundary-value problems. Perhaps one reason for this is the generally-

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accepted view that a symmetry of a boundary-value problem must satisfy three criteria [1]. Namely, it must be

(1) a symmetry of the governing differential equation;
(2) a smooth bijective mapping of the domain to itself;
(3) a mapping of the set of boundary data to itself.

The procedure for finding Lie point symmetries of a given differential equation is well known (see [2–4] for an introduction and [1,5,6] for a more detailed description). This procedure has been implemented within many computer algebra packages [7]. Once the Lie point symmetries are known, they can be used to construct the discrete point symmetries [8]. Thus it is commonly quite easy to find all point symmetries that satisfy condition (1). The difficulty is that the extra conditions (2) and (3) may not be satisfied by any of these symmetries; in this case it seems that the boundary-value problem has no point symmetries. Even if some symmetries satisfy conditions (2) and (3), they are not usually sufficient in number to yield a complete solution of the given boundary-value problem.

So far, research has focused on the alternative problem of classifying all initial or boundary conditions that are consistent with symmetries of a particular differential equation. For integrable partial differential equations (PDEs), a test for consistency between higher symmetries and boundary conditions has been developed recently [9].

Solvable initial-value problems (IVPs) for evolution equations with two independent variables can be catalogued as follows [10]. First, conditional symmetries of the PDE are sought, and the invariance condition leads to an ordinary differential equation (ODE) for the dependent variable. Then consistency conditions are used to find out which initial conditions reduce the original IVP to a Cauchy problem for a system of ODEs.

For linear (or linearizable) PDEs, some IVPs can be solved by solving a simpler IVP [11,12]. The idea is to start with a solvable IVP, and then to use the symmetries associated with linear superposition to construct a hierarchy of related IVPs. In this way, one can construct a catalogue of problems that are solvable for that particular PDE. At present, there is no general procedure for dealing with a given IVP that lacks sufficient symmetries satisfying (1)–(3) and is neither linear nor evolutionary.

It is worth examining the conditions (1)–(3) with a critical eye; are they really necessary? Clearly, conditions (2) and (3) are necessary, for otherwise the boundary-value problem would be mapped to a different boundary-value problem. However, given any Lie point symmetry generator

$$\tilde{X} = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

(where $\partial_x = \partial/\partial x$, etc.) there is an equivalent dynamical symmetry generator

$$X = Q(x, y, y')\partial_y, \quad \text{where } Q(x, y, y') = \eta(x, y) - y'\xi(x, y).$$

The independent variable x is invariant under the symmetries generated by X, so condition (2) is automatically satisfied. From now on, we shall write all generators in the evolutionary form X. However, we restrict attention to dynamical symmetries that are equivalent to point symmetries, so that the characteristic $Q(x, y, y')$ is linear in y'. We shall refer to such symmetries as point-equivalent symmetries.