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A supplement to precise asymptotics in the law of the iterated logarithm

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Abstract

Let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of real-valued i.i.d. random variables with $E(X) = 0$ and $E(X^2) = 1$, and set $S_n = \sum_{i=1}^{n} X_i$, $n \geqslant 1$. This paper studies the precise asymptotics in the law of the iterated logarithm. For example, using a result on convergence rates for probabilities of moderate deviations for $\{S_n;\ n \geqslant 1\}$ obtained by Li et al. [Internat. J. Math. Math. Sci. 15 (1992) 481–497], we prove that, for every $b \in (-1/2, 1]$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{(2b+1)/2} \sum_{n \geqslant 3} \frac{(\log\log n)^b}{n} P\big(|S_n| \geqslant \sigma_n \sqrt{(2+\varepsilon) n \log\log n} + a_n\big)$$

$$= e^{-\sqrt{2}\gamma} 2^b \sqrt{2/\pi}\, \Gamma\big(b + (1/2)\big),$$

whenever $\lim_{n\to\infty} \big(\frac{\log\log n}{n}\big)^{1/2} a_n = \gamma\ \in [-\infty, \infty]$, where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$, $s > 0$, $\sigma^2(t) = E(X^2 I(|X| < \sqrt{t})) - (E(XI(|X| < \sqrt{t})))^2$, $t \geqslant 0$, and $\sigma_n^2 = \sigma^2(n \log\log n)$, $n \geqslant 3$. This result generalizes and improves Theorem 2.8 of Li et al. [Internat. J. Math. Math. Sci. 15 (1992) 481–497] and Theorem 1 of Gut and Spătaru [Ann. Probab. 28 (2000) 1870–1883].

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1. Introduction

Throughout this paper, let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of real-valued independent and identically distributed (i.i.d.) random variables and, as usual, let $S_n = \sum_{i=1}^{n} X_i$, $n \geqslant 1$, denote their partial sums. One of the most striking and fundamental results of probability theory is the renowned law of the iterated logarithm due to Hartman and Wintner [10] for a sequence of i.i.d. random variables. This result, which describes the almost sure (a.s.) asymptotic fluctuation behavior of the partial sums, is stated as follows.

Theorem A. *Let $S_n = \sum_{i=1}^{n} X_i$, $n \geqslant 1$, where $\{X, X_n;\ n \geqslant 1\}$ is a sequence of i.i.d. random variables. If*

$$E(X) = 0 \quad \textit{and} \quad E(X^2) = 1, \tag{1.1}$$

then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1 \quad \textit{a.s.} \quad \textit{and} \quad \liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1 \quad \textit{a.s.} \tag{1.2}$$

Alternative proofs of the Hartman–Wintner [10] law of the iterated logarithm were discovered by Strassen [19], Heyde [12], Egorov [4], Teicher [21], Csörgő and Révész [2, p. 119], and de Acosta [1]. Strassen [20] proved the converse: (1.2) $\Rightarrow$ (1.1). Substantially simpler proofs of Strassen's converse were obtained by Feller [5], Heyde [11], and Steiger and Zaremba [18]. Martikainen [15], Rosalsky [17], and Pruitt [16] simultaneously and independently obtained a "one-sided" converse to the Hartman–Wintner [10] law of the iterated logarithm. Specifically, they proved that each half of (1.2) *individually* implies (1.1).

The following three statements, related to the Hartman–Wintner [10] law of the iterated logarithm, are known to be equivalent:

$$E(X) = 0 \quad \text{and} \quad E(X^2) = 1, \tag{1.3}$$

$$\sum_{n\geqslant 3} \frac{1}{n} P\big(|S_n| \geqslant \sqrt{(2+\varepsilon)n\log\log n}\,\big) \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } -2 < \varepsilon < 0, \end{cases} \tag{1.4}$$

$$\sum_{n\geqslant 3} \frac{\log\log n}{n} P\big(|S_n| \geqslant \sqrt{(2+\varepsilon)n\log\log n}\,\big) \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } -2 < \varepsilon < 0. \end{cases} \tag{1.5}$$

The implication "(1.3) $\Rightarrow$ (1.4)" should be due to Davis [3, Theorem 4] which was remedied by Li et al. [14, Corollary 2.3]. The equivalence "(1.3) $\Leftrightarrow$ (1.5)" was established by Li [13, Corollary 2.2]. For the implication "(1.4) $\Rightarrow$ (1.3)," see Gut [8, Theorem 6.2]. Necessary and sufficient conditions for (1.4) and (1.5), respectively, in a Banach space setting were obtained by Li [13].