Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 302 (2005) 166–179

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

On variable-step relaxed projection algorithm for variational inequalities ☆

Qingzhi Yang [a]

[a] *School of Mathematics and LPMC, Nankai University, Tianjin 300071, PR China*

Received 5 May 2004

Available online 2 October 2004

Submitted by M.A. Noor

Abstract

Projection algorithms are practically useful for solving variational inequalities (VI). However some among them require the knowledge related to VI in advance, such as Lipschitz constant. Usually it is impossible in practice. This paper studies the variable-step basic projection algorithm and its relaxed version under weakly co-coercive condition. The algorithms discussed need not know constant/function associated with the co-coercivity or weak co-coercivity and the step-size is varied from one iteration to the next. Under certain conditions the convergence of the variable-step basic projection algorithm is established. For the practical consideration, we also give the relaxed version of this algorithm, in which the projection onto a closed convex set is replaced by another projection at each iteration and latter is easy to calculate. The convergence of relaxed scheme is also obtained under certain assumptions. Finally we apply these two algorithms to the Split Feasibility Problem (SFP).

Keywords: Variational inequality; Weakly co-coercive; Projection; Relaxation; SFP

☆ This work is supported by the National Natural Science Foundation of China, Grant 10171055.
E-mail address: qz-yang@nankai.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.07.048

1. Introduction

Variational inequality problem is to find an $x \in C$ such that

$$\langle f(x), y - x \rangle \geqslant 0, \quad \text{for all } y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset in R^n and $f(x)$ is a mapping from R^n to itself and $\langle\, , \rangle$ denotes the inner product. Due to their wide applications in various fields, variational inequalities (VI) have received great attention since 1970s, and achieved the fruitful results in the theory as well as applications. The interested reader may consult a two-volume monograph by F. Facchinei and J.S. Pang, which presents a comprehensive, state-of-the art treatment of the finite dimensional variational inequality and complementarity problems [1].

Numerous algorithms for VI have been proposed. Among them, projection-type methods are simple in form and useful in practice provided the projection onto C is easy to calculate. Various projection algorithms, such as basic projection algorithm, extragradient projection algorithm and hyperplane projection algorithm, have been designed to solve the different class of VIs (see, e.g., [1,2,4,6,12,14–21] and references therein). Generally speaking, each projection algorithm is confined in certain class of VIs so that the convergence of the algorithm can be guaranteed. So one usually hopes that an effective algorithm may be used in a broader scope if it possible. For example, consider the following basic projection algorithm with a constant step

$$x^{k+1} = P_C\big[x^k - \gamma f\big(x^k\big)\big]. \tag{1.2}$$

In the early stages of studying projection methods, $f(x)$ was required to be strongly monotone and Lipschitz continuous with small γ for the convergence of the algorithm. Later, this condition is weaken to only require the co-coercivity of $f(x)$ (see [1, p. 1111], [9,13]) while γ is chosen in an interval related to the co-coercive constant. However, in practice we usually cannot get the knowledge of that constant in advance.

Therefore some algorithms are specially proposed so that they may be performed without the requirement of prior knowledge related to $f(x)$. Generally the step-size in this class of algorithms is varied from one iteration to the next in order to guarantee the convergence of an algorithm. In this paper we study the variable-step basic projection algorithm solving VI, which has the following form:

$$x^{k+1} = P_C\big[x^k - \gamma_k f\big(x^k\big)\big]. \tag{1.3}$$

Though in ([1, Algorithm 12.1.4], [20,21]) a variable-step basic projection algorithm is given, actually there variable steps depend on the co-coercive constant. So in this paper we focus on a variable-step basic projection algorithm, where the variable steps are independent of the co-coercive constant. The first algorithm discussed below was actually proposed by Auslender [4] in 1970s. Later Fukushima [6] gave its relaxed version for practical purpose. Both Auslender and Fukushima assumed the strong monotonicity of $f(x)$ for the convergence of their algorithms. This is a strict limit. We establish the convergence of Auslender's algorithm as well as Fukushima's algorithm under weaker conditions. Mainly the strong monotonicity is replaced by the weak co-coercivity, whose a special case is