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Some reduction and transformation formulas for the Appell hypergeometric function F_2

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Abstract

An integral representation of the Appell series $F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$ is used here to obtain several finite-sum expansions in terms of the less cumbersome hypergeometric functions ${}_2F_1$ and ${}_3F_2$. In the case when the parameters σ, α_1, α_2, β_1, and β_2 are all positive integers, some of our results may be seen as a generalization of the finite-sum expansions of the Appell series F_1 obtained by Cuyt et al. [J. Comput. Appl. Math. 5 (1999) 213–219]. Other important consequences (as well as potential applications) of our results are discussed and a listing of some useful reduction (and transformation) formulas is provided.

Keywords: Multiple hypergeometric functions; Appell series; Integral representations; Singular potentials; Gordon's integral; Gauss hypergeometric function; Clausen hypergeometric function; Kummer (or confluent) hypergeometric function; Reduction and transformation formulas; Singular operators; Dilogarithm function; Digamma (or Psi) function

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1. Introduction and definitions

The Appell hypergeometric series

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$$

arises frequently in various physical and chemical applications (cf. [2,3,5,6,8,11]). It is defined by

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\sigma)_{m+n}(\alpha_1)_m(\alpha_2)_n}{(\beta_1)_m(\beta_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$
$$\big(|x| + |y| < 1;\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\big), \tag{1.1}$$

where $(\lambda)_k$ denotes the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & \big(k = 0;\ \lambda \in \mathbb{C} \setminus \{0\}\big), \\ \lambda(\lambda + 1)(\lambda + 2)\cdots(\lambda + k - 1) & (k \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

$\mathbb{N}$ being the set of *positive* integers.

Recently, Cuyt et al. [1] obtained a finite algebraic sum representation of the Appell hypergeometric series F_1 defined by

$$F_1(a, b, b'; c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \quad \big(\max\big\{|x|, |y|\big\} < 1\big) \tag{1.2}$$

in the case when the parameters a, b, b', and c are *positive* integers with $c > a$. Their results can be summarized for the simplest case as follows [1, p. 215, Theorem 2.1a, b, and c]:

For any $s, t \in N_0 := \mathbb{N} \cup \{0\}$ *and* $x \neq y$,

$$\begin{aligned} F_1(1, s + 1, t + 1; 2; x, y) = {}& \frac{(-x)^t y^s}{(y - x)^{s+t+1}}\binom{s + t}{s}\big[\ln(1 - x) - \ln(1 - y)\big] \\ & - \sum_{j=0}^{t-1}\binom{j + s}{s}\frac{(-x)^j y^s[1 - (1 - y)^{j-t}]}{(y - x)^{s+j+1}(t - j)} \\ & - \sum_{k=0}^{s-1}\binom{k + t}{t}\frac{x^t(-y)^k[1 - (1 - x)^{k-s}]}{(x - y)^{t+k+1}(s - k)} \end{aligned}$$
$$\big(\max\big\{|x|, |y|\big\} < 1\big). \tag{1.3}$$

Motivated by the aforementioned investigation by Cuyt et al. [1], we extend their results to obtain a finite algebraic sum representation of Appell's *second* series

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$$

in the case when the parameters σ, α_1, α_2, β_1, and β_2 are *positive* integers. We show that their results are indeed special cases of our representation for F_2. As intermediate steps, we derive a number of reduction and transformation formulas for F_2. The paper is organized as follows: In Section 2, we develop reduction formulas for F_2 in terms of the Gauss hypergeometric function ${}_2F_1$ and the Clausen hypergeometric function ${}_3F_2$. We then show that the derived form of F_2 reduces to the known expression (1.3) for F_1. In Section 3, we extend these results to obtain finite-sum representations of F_2 with *nonnegative* parameters.