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J. Math. Anal. Appl. 302 (2005) 217–229
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On a third-order multi-point boundary value problem at resonance ☆

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Received 9 June 2003
Available online 22 September 2004
Submitted by J. Mawhin

Abstract

In this paper, we prove some existence results for a third order multi-point boundary value problem at resonance. Our method is based upon the coincidence degree theory of Mawhin.

Keywords: Multi-point boundary value problem; Resonance; Coincidence degree theory; Third-order differential equations

1. Introduction

This paper deals with the following third-order ordinary differential equation:

$$x'''(t) = f\big(t, x(t), x'(t), x''(t)\big) + e(t), \quad t \in (0, 1), \tag{1.1}$$

with the following boundary value conditions:

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(0) = 0, \qquad x(1) = \beta x(\eta). \tag{1.2}$$

☆ Sponsored by the National Natural Science Foundation of China (No. 10371006).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.08.012

Here $f:[0,1]\times R^3\to R$ is a continuous function, $e\in L^1[0,1]$, α_i $(1\leqslant i\leqslant m-2)\in R$, $\beta\geqslant 0$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, and $\eta\in(0,1)$.

Similarly in [1,2], for certain boundary condition case such that the linear operator $Lx=x'''(t)=0$, defined in a suitable Banach space, is invertible, the so-called non-resonance case. Otherwise, the so-called resonance case.

For the non-resonance case, we refer to see [3,10] and the references therein.

For the resonance case, the boundary value problem is approached in several ways. Such as, Ma [7] studied existence and multiplicity results for the boundary value problem

$$x'''+k^2x'+g(x,x')=p(t),\qquad x'(0)=x'(\pi)=x(\eta)=0. \tag{1.3}$$

by combining the Lyapunov–Schmit procedure with the continuum theory for O-epi maps. In the case $k=1$, the solvability of (1.3) has been considered by Nagle and Pothoven [9] under the condition that g is bounded on one side. But the more classical method is to decompose the space in the form of a direct sum of subspaces, one of which is $\operatorname{Ker} L$, and then to work with the corresponding projections on these spaces. For instance, Feng [1], Gupta [4,5], and Liu and Yu [6] used this method to study the existence results for some second order multi-point boundary value problems at resonance case.

Inspired by the work of the above papers, in the present article, we use the coincidence degree theory of Mawhin [8] to discuss the existence of solution for third-order multi-point BVP (1.1), (1.2) at resonance case, and establish some existence theorems under nonlinear growth restriction of f.

2. Existence results

First we present some preliminaries needed to understand how the fixed point result of Mawhin [8] is concerned.

Let Y, Z be real Banach spaces and let $L:\operatorname{dom} L\subset Y\to Z$ be a linear operator which is Fredholm map of index zero and $P:Y\to Y$, $Q:Z\to Z$ be continuous projectors such that $\operatorname{Im} P=\operatorname{Ker} L$, $\operatorname{Ker} Q=\operatorname{Im} L$ and $Y=\operatorname{Ker} L\oplus\operatorname{Ker} P$, $Z=\operatorname{Im} L\oplus\operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L\cap\operatorname{Ker} P}:\operatorname{dom} L\cap\operatorname{Ker} P\to\operatorname{Im} L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\operatorname{dom} L\cap\Omega\neq\emptyset$, the map $N:Y\to Z$ is said to be L-compact on $\bar\Omega$ if the map $QN(\bar\Omega)$ is bounded and $K_P(I-Q)N:\bar\Omega\to Y$ is compact. For more details we refer the reader to the lecture notes of Mawhin [8].

To obtain our existence results we use the following fixed point theorem of Mawhin [8].

Theorem 2.1. *Let L be a Fredholm operator of index zero and let N be L-compact on $\bar\Omega$. Assume that the following conditions are satisfied*:

(i) $Lx\neq\lambda Nx$ *for every* $(x,\lambda)\in[(\operatorname{dom} L\setminus\operatorname{Ker} L)\cap\partial\Omega]\times(0,1)$.
(ii) $Nx\notin\operatorname{Im} L$ *for every* $x\in\operatorname{Ker} L\cap\partial\Omega$.
(iii) $\deg(QN|_{\operatorname{Ker} L},\Omega\cap\ker L,0)\neq 0$, *where* $Q:Z\to Z$ *is a projection as above with* $\operatorname{Im} L=\operatorname{Ker} Q$.

Then the equation $Lx=Nx$ has at least one solution in $\operatorname{dom} L\cap\bar\Omega$.