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Does distribution theory contain means for extending Poincaré–Bendixson theory?

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Abstract

We use the theory of distributions to extend the Poincaré–Bendixson theorem and the Bendixson criterion to piecewise Lipschitz continuous system possessing unique and continuous solutions. We demonstrate the use of these extensions by several examples that have recently appeared in the literature.

Keywords: Poincaré–Bendixson theorem; Distributions; Piecewise linear systems

1. Introduction

The study of piecewise linear systems has been essential for applications like control theory, electronics and automatic navigation systems, during the past decades. The formulation of a rich and satisfactory theory for such systems is of utmost importance. Yet, only a few attempts to treat such systems in a general and abstract mathematical setting has been made. Many papers that have appeared quite recently contain, for instance, explicit calculations in specific systems in order to estimate position and number of limit cycles in two-dimensional cases [4,6–9]. In this paper we suggest a new approach based on distrib-

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ution theory [5] in the two-dimensional case. We do not intend to make precise statements regarding the most general cases here, but our approach cover most cases that appear in the application areas mentioned above including optimal foraging theory in mathematical ecology [1]. Our paper is organized as follows. We formulate our two-dimensional setting and our generalized two basic theorems in Section 2. These two generalizations are the essence of the paper. In Section 3 we demonstrate the use of those theorems in several classical examples that contains many difficulties connected to differential equations with discontinuous right-hand sides. Any satisfactory theory must fully explain these examples. We attach figures to most of the examples giving the reader a rapid understanding of what ought to be explained. In Section 4, we give a short summary of our results and list some of their main implications.

2. Our settings and main theorems

We shall work with planar systems with discontinuous right-hand sides throughout this paper. We restrict the properties of the systems under consideration by four major assumptions. The purpose of this paper is to give a presentation of some new ideas, and for simplicity and clarity we do not formulate these ideas in their most general context.

We consider a planar autonomous system

$$\dot{x} = f(x). \tag{1}$$

- (A1) Ω is an open domain in R^2, divided into a finite number of open sub-domains Ω_i, such that $\bigcup \bar{\Omega}_i = \bar{\Omega}$.
- (A2) If $\bar{\Omega}_i$ and $\bar{\Omega}_j$ are not disjoint and $i \neq j$, then $\bar{\Omega}_i \cap \bar{\Omega}_j = \Gamma_{ij}$, where Γ_{ij} (joint boundaries) are piecewise smooth.
- (A3) f is Lipschitz in all sub-domains Ω_i and possibly discontinuous along Γ_{ij} (also called discontinuity curves).
- (A4) The vector field f defines a direction in each point in Ω. In particular, at every point of Γ_{ij} the vector field $f(x)$ specifies into which Ω_i the flow is directed.

The conditions (A3) and (A4) implies that the differential equation (1) has unique, continuous and piecewise smooth solutions in Ω. Note that (A4) gives strong restrictions on the possible discontinuities. In terms of Filippov [3] there are three kinds of sliding modes. We only allow transversal sliding mode, that is: the vector field is directed from one side to the other at the discontinuity curves. The solutions will pass the discontinuity curves in the field direction and we have uniqueness of solutions there. Attracting and repulsion sliding mode will be excluded.

Theorem 1 (Extension of the Poincaré–Bendixson theorem). *Consider the planar autonomous system* (1). *Let the conditions* (A1)–(A4) *be satisfied and let f be bounded in Ω. Suppose that K is a compact region in Ω, containing no fixed points of* (1). *If all solutions of* (1) *is in K, for all $t \geqslant t_0$, then* (1) *has a closed orbit in K.*