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Semiconcavity results for constrained optimal control problems in a half-space

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Abstract

In this paper we give semiconcavity results for the value function of some constrained optimal control problems with infinite horizon in a half-space. In particular, we assume that the control space is the l^1-ball or the l^∞-ball in $\mathbb{R}^n$.

Keywords: Optimal control problems; State constraints; Dynamic programming; Semiconcavity

1. Introduction

We are interested in studying semiconcavity results for the value function of optimal control problems with state constraints. We recall that, roughly speaking, a semiconcave function is a function that can be locally represented as the sum of a concave function plus a smooth one. Therefore, semiconcave functions share many differentiability properties of concave functions. For example, they are twice differentiable almost everywhere and possess a nonempty superdifferential at any point. This fact can be used to derive necessary and sufficient optimality conditions (see [7,10]). Moreover, sharp Hausdorff estimates from above and below are available for the singular set of a semiconcave function; see [1,2].

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Semiconcavity property is quite natural in control theory without state constraints. Generally, the value function is a nonsmooth function, viscosity solution of an associated Hamilton–Jacobi equation. This means that suitable inequalities hold for the super- and sub-differential at any point (see [5,6,14]).

For several optimal control problems without state constraints semiconcavity results have been proved; see, e.g., [5,7,10]. On the other hand, proving a semiconcavity estimate for the value function of constrained problems is a harder task, even for simple models. We consider some special cases of optimal control problems and we do not know other semiconcavity results for constrained problems except two results obtained by P. Cannarsa and the author: the first in the one-dimensional case [8], the latter for a simple exit time problem [9].

It is well known that the value function of an optimal control problem with state constrains can be characterized as the unique viscosity solution of a suitable Hamilton–Jacobi–Bellman equation; see [12,16]. Note that, in general, we do not expect constrained viscosity solutions of Hamilton–Jacobi equations to be semiconcave. Indeed, in [3] is proved that solutions of a large class of Hamilton–Jacobi equations with state constraints boundary conditions are convex.

Let $x = (x_1, \dots, x_n)$ be a generic vector in $\mathbb{R}^n$, we denote by $\mathbb{R}^n_+$ the set

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \colon\ x_n \geqslant 0\}$$

and consider the problem

$$\begin{cases} \dot{y}(t) = u(t), & t > 0, \\ y(0) = x \in \mathbb{R}^n_+. \end{cases} \tag{1.1}$$

Here u is a *control*—i.e. a measurable function taking values in a given closed set $U \subset \mathbb{R}^n_+$ (the *control space*). We will denote by $y^u_x(t)$ the solution of (1.1), and say that $y^u_x(t)$ is the trajectory starting at x with control u.

We say that a control u is *admissible* at a point $x \in \mathbb{R}^n_+$ if the corresponding trajectory remains in $\mathbb{R}^n_+$ for all positive times and define the set of admissible controls at x:

$$\mathcal{A}_x = \bigl\{u : [0,\infty) \to U \colon\ y^u_x(t) \in \mathbb{R}^n_+,\ \forall t \geqslant 0\bigr\}, \quad \forall x \in \mathbb{R}^n_+.$$

Consider the cost functional

$$J(x,u) = \int_0^\infty e^{-\lambda t} L\bigl(y^u_x(t)\bigr)\, dt, \quad x \in \mathbb{R}^n_+,$$

where the function $L : \mathbb{R}^n_+ \to \mathbb{R}$ is a *running cost* verifying suitable assumptions and λ, the *discount factor,* is a positive number. The *value function* of this optimal control problem is defined as

$$v(x) = \inf_{u \in \mathcal{A}_x} J(x,u). \tag{1.2}$$

A control $\hat{u}$ is *optimal* for $x \in \mathbb{R}^n_+$ iff $v(x) = J(x,\hat{u})$. In this case $y^{\hat{u}}_x$ is said to be an optimal trajectory.

In order to analyze the regularity properties of the value function v a crucial tool is given by the dynamic programming principle that, in the present case, can be stated as follows.