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Properties of symmetric matrices

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Abstract

We find the extreme points and the smooth points of the unit ball of the Banach space $\wp_n$ of all real symmetric $(n \times n)$-matrices.
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1. Introduction

Let B_X and ∂B_X denote the closed unit ball and the unit sphere of a real Banach space X, respectively. A point $x \in B_X$ is said to be an extreme point of B_X if x lies on no segment both of whose end points lie in B_X and are different from x. We denote the set of all extreme points of B_X by $E(B_X)$. A point $x \in B_X$ is said to be a smooth point of B_X if there exists a unique Λ in the dual space X^* of X such that $\|\Lambda\| = \Lambda x = 1$. Geometrically, a point $x \in B_X$ is a smooth point if there exists a unique supporting tangent plane at x. We denote the set of all smooth points of B_X by $S(B_X)$. Recall that a continuous m-homogeneous polynomial on a real Banach space X is a function $p : X \to \mathbb{R}$ for which there exists a necessarily unique continuous, symmetric m-linear form $F : X^m \to \mathbb{R}$ with the property that $p(x) = F(x, \dots, x)$ for every $x \in X$. We denote by $\wp({}^m X)$ the Banach

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space of all continuous m-homogeneous polynomials on X, where the norm is given by $\|p\| = \sup\{|p(x)|: \|x\| \leqslant 1\}$ [2]. Then, for a positive integer n,

$$\wp\left({}^2\mathbb{R}^n\right) = \left\{p: \begin{array}{l} p(x) = x^t P x, \text{ where } P \text{ is a real symmetric} \\ (n \times n)\text{-matrix and } x \in \mathbb{R}^n \end{array}\right\}.$$

Moreover, by Spectral theorem, for $p \in \wp({}^2\mathbb{R}^n)$ we can write

$$p(x) = x^t U^t \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U x,$$

where $U \in O(n)$, the n-dimensional orthogonal group. Thus if $\|x\| \leqslant 1$, then we have

$$\left| x^t U^t \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U x \right| \leqslant \max_{1 \leqslant i \leqslant n} |\lambda_i|.$$

Also for a 2-homogeneous polynomial p with $p(x) = x^t (a_{ij})_{n\times n} x$ on $\mathbb{R}^n$, it is a necessary condition that $|a_{ij}| \leqslant 1$ in order for p to have norm 1.

Hence we get

$$\|p\| = \text{the maximum absolute value of eigenvalues of } P.$$

Now we denote the space of all real symmetric $(n \times n)$-matrices by $\wp_n$. From the above mentioned remark we see that $\wp_n$ is a Banach space under the norm $\|P\|$ = the maximum absolute value of eigenvalues of P.

The purpose of this paper is to classify the extreme points and the smooth points of $B_{\wp_n}$ by finding out relevant properties of symmetric matrices. Indeed, we will prove:

Theorem 1.1. *$P \in E(B_{\wp_n})$ if and only if the absolute values of eigenvalues of P are identically* 1, *i.e., the set of all extreme points of $B_{\wp_n}$ is the following*:

$$\left\{ U^t \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U\colon\ U \in O(n),\ |\lambda_i| = 1,\ \textit{and}\ 1 \leqslant i \leqslant n \right\}.$$

Theorem 1.2. *$P \in S(B_{\wp_n})$ if and only if P has exactly one eigenvalue whose absolute value is* 1.

To mention some of the previous results related with this, R. Grzaslewicz and K. John characterized the extreme points of the unit ball of all bilinear operators on the 2-dimensional real Hilbert space ℓ_2^2 [5]. Z. Hu and B. Lin obtain a result on extremal structure of the unit ball of $L^p(\mu, X)^*$ [6]. R.M. Aron and M. Klimek characterized the extreme points of the unit ball in the space of all quadratic polynomials of one real variable with supremum norm over the interval $[-1, 1]$ [1]. Y.S. Choi and S. Kim characterized the extreme points and smooth points of $B_{\wp({}^2\ell_2^2)}$ [4]. Therefore, our Theorems 1.1 and 1.2 generalize and extend the result of [4] to the n-dimensional real Hilbert space $\ell_2^n = \mathbb{R}^n$.

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