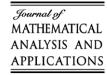


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### Properties of symmetric matrices

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#### **Abstract**

We find the extreme points and the smooth points of the unit ball of the Banach space  $\wp_n$  of all real symmetric  $(n \times n)$ -matrices.

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#### 1. Introduction

Let  $B_X$  and  $\partial B_X$  denote the closed unit ball and the unit sphere of a real Banach space X, respectively. A point  $x \in B_X$  is said to be an extreme point of  $B_X$  if x lies on no segment both of whose end points lie in  $B_X$  and are different from x. We denote the set of all extreme points of  $B_X$  by  $E(B_X)$ . A point  $x \in B_X$  is said to be a smooth point of  $B_X$  if there exists a unique A in the dual space  $X^*$  of X such that ||A|| = Ax = 1. Geometrically, a point  $x \in B_X$  is a smooth point if there exists a unique supporting tangent plane at x. We denote the set of all smooth points of  $B_X$  by  $S(B_X)$ . Recall that a continuous m-homogeneous polynomial on a real Banach space X is a function  $p: X \to \mathbb{R}$  for which there exists a necessarily unique continuous, symmetric m-linear form  $F: X^m \to \mathbb{R}$  with the property that p(x) = F(x, ..., x) for every  $x \in X$ . We denote by  $\wp(m^m X)$  the Banach

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space of all continuous *m*-homogeneous polynomials on *X*, where the norm is given by  $||p|| = \sup\{|p(x)|: ||x|| \le 1\}$  [2]. Then, for a positive integer *n*,

$$\wp(^{2}\mathbb{R}^{n}) = \left\{ p \colon \begin{array}{ll} p(x) = x^{t} P x, \text{ where } P \text{ is a real symmetric} \\ (n \times n)\text{-matrix and } x \in \mathbb{R}^{n} \end{array} \right\}.$$

Moreover, by Spectral theorem, for  $p \in \wp(^2\mathbb{R}^n)$  we can write

$$p(x) = x^{t} U^{t} \begin{pmatrix} \lambda_{1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n} \end{pmatrix} U x,$$

where  $U \in O(n)$ , the *n*-dimensional orthogonal group. Thus if  $||x|| \le 1$ , then we have

$$\begin{vmatrix} x^t U^t \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} Ux \end{vmatrix} \leqslant \max_{1 \leqslant i \leqslant n} |\lambda_i|.$$

Also for a 2-homogeneous polynomial p with  $p(x) = x^t (a_{ij})_{n \times n} x$  on  $\mathbb{R}^n$ , it is a necessary condition that  $|a_{ij}| \leq 1$  in order for p to have norm 1.

Hence we get

||p|| = the maximum absolute value of eigenvalues of P.

Now we denote the space of all real symmetric  $(n \times n)$ -matrices by  $\wp_n$ . From the above mentioned remark we see that  $\wp_n$  is a Banach space under the norm ||P|| = the maximum absolute value of eigenvalues of P.

The purpose of this paper is to classify the extreme points and the smooth points of  $B_{\wp_n}$  by finding out relevant properties of symmetric matrices. Indeed, we will prove:

**Theorem 1.1.**  $P \in E(B_{\wp_n})$  if and only if the absolute values of eigenvalues of P are identically 1, i.e., the set of all extreme points of  $B_{\wp_n}$  is the following:

$$\left\{ U^t \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U \colon U \in O(n), \ |\lambda_i| = 1, \ and \ 1 \leqslant i \leqslant n \right\}.$$

**Theorem 1.2.**  $P \in S(B_{\wp_n})$  if and only if P has exactly one eigenvalue whose absolute value is 1.

To mention some of the previous results related with this, R. Grzaslewicz and K. John characterized the extreme points of the unit ball of all bilinear operators on the 2-dimensional real Hilbert space  $\ell_2^2$  [5]. Z. Hu and B. Lin obtain a result on extremal structure of the unit ball of  $L^p(\mu,X)^*$  [6]. R.M. Aron and M. Klimek characterized the extreme points of the unit ball in the space of all quadratic polynomials of one real variable with supremum norm over the interval [-1,1] [1]. Y.S. Choi and S. Kim characterized the extreme points and smooth points of  $B_{\wp(2\ell_2^2)}$  [4]. Therefore, our Theorems 1.1 and 1.2 generalize and extend the result of [4] to the n-dimensional real Hilbert space  $\ell_2^n = \mathbb{R}^n$ .

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