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Spectral synthesis on torsion groups ☆

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Abstract

Spectral synthesis on Abelian torsion groups is proved.
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In this paper $\mathbb{C}$ denotes the set of complex numbers. If G is an Abelian group then $\mathbb{C}^G$ denotes the locally convex topological vector space of all complex valued functions defined on G, equipped with the pointwise operations and the product topology. The dual of $\mathbb{C}^G$ can be identified with $\mathcal{M}_c(G)$, the space of all finitely supported complex measures on G. This space is also identified with the set of all finitely supported complex valued functions on G in the following obvious way. If the point mass concentrated at the element g is denoted by δ_g, then each measure x in $\mathcal{M}_c(G)$ has a unique representation in the form

$$x = \sum_{g \in G} x(g)\delta_g$$

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with some finitely supported function $x : G \to \mathbb{C}$. "Identification" means that we use the same letter x for both the measure and the representing function. In this sense δ_g is the characteristic function of the singleton $\{g\}$. The pairing between $\mathbb{C}^G$ and $\mathcal{M}_c(G)$ is given by the formula

$$\langle x, f\rangle = \sum_{g\in G} x(g)\overline{f(g)}.$$

Convolution on $\mathcal{M}_c(G)$ is defined in the usual way by putting

$$x * y(g) = \sum_{h\in G} x(h)y(gh^{-1})$$

for any x, y in $\mathcal{M}_c(G)$ and g in G. This convolution converts the linear space $\mathcal{M}_c(G)$ into a commutative algebra with unit δ_1. One realizes immediately that the algebra $\mathcal{M}_c(G)$ is identical with the finite group algebra of G. Hence we can use the alternative notation $\mathbb{C}G$ for $\mathcal{M}_c(G)$, which may be more familiar for algebraists. In this sense we can consider G as a subset of $\mathbb{C}G$ by identifying the group element g with the corresponding measure δ_g.

Homomorphisms of G into the additive group of complex numbers, or into the multiplicative group of nonzero complex numbers are called *additive*, or *exponential functions*, respectively. Bounded exponential functions are exactly the *characters* of G. A function of the form $g \mapsto P(a_1(g), a_2(g), \ldots, a_n(g))$ on G is called a *polynomial*, if P is a complex polynomial in n variables and a_k is additive for $k = 1, 2, \ldots, n$. A complex valued function on G is called an *exponential monomial* if it is the product of a polynomial and an exponential. Linear combinations of exponential monomials are called *exponential polynomials*. In particular, linear combinations of characters are called *trigonometric polynomials*.

Translation with the element h in G is the operator mapping any function f in $\mathbb{C}^G$ onto its *translate* $\tau_h f$ defined by $\tau_h f(g) = f(gh)$ for any g in G. A subset of $\mathbb{C}^G$ is called *translation invariant* if it contains all translates of its elements. A closed linear subspace of $\mathbb{C}^G$ is called a *variety* on G if it is translation invariant.

For any variety V on G the *annihilator of* V is the set $V^\perp$ of all measures in $\mathbb{C}G$ which vanish on V. Clearly this is an ideal, which is proper if and only if V is nonzero. Similarly, for any ideal I in $\mathbb{C}G$ the *annihilator of* I is the set $I^\perp$ of all functions in $\mathbb{C}^G$, which are annihilated by all measures in I. Clearly this is a variety on G, which is nonzero if and only if I is proper. Moreover, by the Hahn–Banach theorem it is clear that $V^{\perp\perp} = V$ and $I^{\perp\perp} = I$ holds for each variety on G and for any ideal I in $\mathbb{C}G$. For more details see, e.g., [7].

The basic question of spectral analysis on G can be formulated as follows: does any nonzero variety on G contain an exponential function? If so, then we say that *spectral analysis holds on* G. In [8] the second author proved that if G is an Abelian torsion group then the answer is "yes." Recently in [4] M. Laczkovich and G. Székelyhidi have presented a complete characterization of Abelian groups having spectral analysis: it is necessary and sufficient that the torsion free rank of the group is less than the continuum.

Another basic problem concerns spectral synthesis on G: given a nonzero variety on G, do the exponential monomials in this variety span a dense subspace? If so, then we say that *spectral synthesis holds on* G. This is the case, for instance, if G is a finitely generated free Abelian group by a result of M. Lefranc [5]. In [2] R.J. Elliot presented a theorem stating

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