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Bifurcating periodic solutions of wind-driven circulation equations

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Abstract

The existence of bifurcating periodic flows in a quasi-geostrophic mathematical model of wind-driven circulation is investigated. In the model, the Ekman number r and Reynolds number R control the stability of the motion of the fluid. Through rigorous analysis it is proved that when the basic steady-state solution is independent of the Ekman number, then a spectral simplicity condition is sufficient to ensure the existence of periodic solutions branching off the basic steady-state solution as the Ekman number varies across its critical value for constant Reynolds number. When the basic solution is a function of Ekman number, an additional condition is required to ensure periodic solutions.

Keywords: Hopf bifurcation; Periodic solution; Navier–Stokes equation; Bessel-potential space

1. Introduction

Bifurcation analysis plays a crucial role in understanding qualitative changes of flow regimes of oceanic and atmospheric circulation equations. From the view point of numer-

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ical experiments, bifurcation phenomena relating to ocean circulation were examined by Charney and DeVore [2], Veronis [19,20], and Pedlosky [15] to determine the occurrence of multi steady-state solutions, whereas Jiang et al. [9], Jin and Ghil [10] predicted periodic circulatory motions.

Because wind-driven circulation problems embody extremely complex physical mechanisms, to aid understanding it is beneficial to study simplified models (see Ghil and Childress [6], Lions et al. [13,14], and Pedlosky [16]). For example, Pedlosky [16] developed a suitable mathematical model describing mid-latitude wind-driven circulation through a quasi-geostrophic approximation. This model describes the dynamics of circulation flow-driven by a wind stress and influenced by Ekman friction layers, bottom topography, and a β-plane approximation of the Coriolis force.

The purpose of this study is to derive a general Hopf bifurcation theorem of the two-dimensional simplified wind-driven circulation equation (see Pedlosky [16, Eq. (5.2.22)]) which can be expressed in the following dimensionless vorticity formulation:

$$\partial_t \Delta\psi + r\Delta\psi - \frac{1}{R}\Delta^2\psi + \beta\partial_x\psi + J(\psi, \Delta\psi + \eta_B) = \beta \operatorname{curl}\tau \quad \text{in } \Omega \tag{1}$$

together with the free slip boundary condition

$$\psi = 0, \quad \Delta\psi = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Here $J(\psi, \phi) = \partial_x\psi\partial_y\phi - \partial_y\psi\partial_x\phi$ is the advection operator, $\psi = \psi(x, y, t)$ describes a geostrophic stream function, $\tau = (\tau_1(x, y), \tau_2(x, y))$ is a steady-state wind stress applied on the circulation basin Ω. The x-axis points eastwards and the y-axis northwards. The zonal and meridional velocity components u and v are given by

$$u = -\partial_y\psi, \qquad v = \partial_x\psi,$$

and the relative vorticity ζ is defined by $\zeta = \Delta\psi$.

The mathematical formulation (1) is a quasi-geostrophic approximation of the shallow water equation under the effect of a Coriolis force (see [16]). The parameter $\eta_B = \eta_B(x, y)$ is a function measuring the topography height of the bottom of the original fluid domain. Control parameters defining the fluid motion are Reynolds number R, the Ekman number r measuring the effects of friction arising from the top and bottom Ekman layers of the fluid, and the number β is derived from a β-plane approximation of the Coriolis force. The interested reader may find further descriptions of this equation and the theoretical background by consulting Pedlosky [16, pp. 260–261].

The existence of bifurcating periodic solutions branching off a basic steady-state solution of an ordinary differential equation system was first derived by Hopf [7,8]. The Hopf bifurcation theorem of Navier–Stokes equations is well-established under the assumption that the nonreal eigenvalue of the linearized spectral problem satisfies the simplicity condition and a condition relating to the transversal crossing of the imaginary axis at a critical value of Reynolds number (see Yudovich [22], Joseph and Sattinger [11]). Equation (1) is similar in form to the vorticity formulation of Navier–Stokes equations. Thus the Hopf bifurcation theorem derived in [11] is applicable to (1) to ensure the existence of bifurcating periodic solutions as the Reynolds number varies across a critical value R_c for constant Ekman number. However (1) also depends on the Ekman number r and this introduces