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Eigenvalues of $p(x)$-Laplacian Dirichlet problem ☆

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Abstract

This paper studies the eigenvalues of the $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^N and $p(x)$ is a continuous function on $\bar{\Omega}$ such that $p(x) > 1$. We show that Λ, the set of eigenvalues, is a nonempty infinite set such that $\sup \Lambda = +\infty$. We present some sufficient conditions for $\inf \Lambda = 0$ and for $\inf \Lambda > 0$, respectively.

Keywords: $p(x)$-Laplacian; Eigenvalue; Generalized Lebesgue–Sobolev space

1. Introduction

The differential equations and variational problems with $p(x)$-growth conditions arise from nonlinear elasticity theory, electrorheological fluids, etc. (see [1,5,19,23–26]). The study of such problems is a new and interesting topic, some results on this topic can be found in [1–5,7,9–11,13,17,19,23–26].

A typical model of elliptic equations with $p(x)$-growth conditions is

$$-\operatorname{div}\left(\left(\alpha + |\nabla u|^2\right)^{(p(x)-2)/2}\nabla u\right) = f(x, u).$$

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In particular, when $\alpha = 0$, the operator $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, it is a natural generalization of the p-Laplacian (where $p > 1$ is a constant). The $p(x)$-Laplacian possesses more complicated nonlinearity than the p-Laplacian, for example, it is inhomogeneous.

In this paper, we shall consider the eigenvalues of the $p(x)$-Laplacian Dirichlet problem

$$\text{(P)} \quad \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^N, $p : \bar{\Omega} \to R$ is a continuous function and $p(x) > 1$ for $x \in \bar{\Omega}$.

In order to deal with the problem (P), we need some theory of the generalized Lebesgue–Sobolev spaces (see [6,8,12,14,16,18,20]). For convenience, we give a simple description here.

Let

$$L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is a measurable real valued function on } \Omega, \right.$$
$$\left. \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$
$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

We can introduce norms on $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, respectively, as

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 \;\middle|\; \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leqslant 1 \right\}, \quad u \in L^{p(x)}(\Omega),$$

and

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega),$$

then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are both separable and reflexive Banach spaces.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$; we know that $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$.

Definition 1.1. Let $\lambda \in R$ and $u \in W_0^{1,p(x)}(\Omega)$. (u, λ) is called a solution of the problem (P) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx = \lambda \int_\Omega |u|^{p(x)-2}uv\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega). \tag{1.1}$$

If (u, λ) is a solution of (P) and $u \not\equiv 0$, as usual, we call λ and u an eigenvalue and an eigenfunction corresponding to λ of (P), respectively.