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Inheritance of noncompactness of the $\bar{\partial}$-Neumann problem

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Abstract

In this paper we show that noncompactness of the $\bar{\partial}$-Neumann operator on a smooth, bounded, pseudoconvex Reinhardt domain Ω in $\mathbb{C}^2$ implies noncompactness of the $\bar{\partial}$-Neumann operator of higher-dimensional domains fibered over Ω under a suitable size restriction on the fibers.

1. Introduction

It has been an open question whether every smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ with an analytic disk in the boundary necessarily has a noncompact $\bar{\partial}$-Neumann operator N. Some partial results are known. For instance, the answer is affirmative both for domains in $\mathbb{C}^2$ [5] and for convex domains in $\mathbb{C}^n$ [4]. It remains open, for example, whether an analytic disk in the boundary of a complete Reinhardt domain in $\mathbb{C}^3$ necessarily obstructs compactness of N. On the other hand, it is known that in the case of Reinhardt domains, disks in the boundary are the only possible obstructions to compactness of N [5].

In this paper we show the following theorem.

Theorem. *Suppose Ω is a smooth, bounded, pseudoconvex Reinhardt domain in $\mathbb{C}^2$ whose $\bar{\partial}$-Neumann operator N is noncompact* (*on the space of square-integrable* $(0,1)$-*forms*).

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If G is a smooth bounded pseudoconvex in $\mathbb{C}^n$ fibered over Ω and there exists a constant C such that all points (z, w) of G (where $z \in \Omega$, $w \in \mathbb{C}^{n-2}$) satisfy the restriction $\|w\| < Cd(z, b\Omega)^{1/2}$, then G has a noncompact $\bar{\partial}$-Neumann operator N.

In the next section, we introduce the notion of "thick subdomain" and some lemmas that enter into the proof of the theorem.

2. Thick subdomains

Let G be a domain in $\mathbb{C}^n$, and let A be a subdomain of G. If there is a sequence $\{f_j\}$ of holomorphic functions in the unit ball of $L^2(G)$ such that no subsequence of $\{f_j\}$ converges in $L^2(A)$, then A is said to be a *thick subdomain* of G. In other words, A is a thick subdomain of G if the restriction operator $L^2(G) \cap \mathcal{O}(G) \to L^2(A)$ is not a compact operator.

For example, if there is a point p in the boundary of G and a neighborhood U of p in $\mathbb{C}^n$ such that $A \cap U = G \cap U$, then A is a thick subdomain of G. For instance, let E be the unit disk $\{z\colon |z| < 1\}$ in $\mathbb{C}$ and let A be $\{(x, y) \in E\colon 0 < x < 1$ and $0 < y < (1-x)^p\}$, where $p > 0$. Then A is a thick subdomain of E if (and only if) $p \leqslant 1$. One can easily check this by taking the sequence of holomorphic functions $\{f_j\}$ to be the sequence of normalized Bergman kernel functions $\{K_E(z, p_j)/\sqrt{K_E(p_j, p_j)}\}$, where the sequence $\{p_j\}$ approaches the point $(1, 0)$.

We recall the definition of the Bergman kernel function. Let $H(D)$ denote the space of square-integrable holomorphic functions on a domain D in $\mathbb{C}^n$. By the Riesz representation theorem, for each fixed point w in D there is a unique element of $H(D)$, denoted by $K_D(\cdot, w)$, such that

$$f(w) = \big(f, K_D(\cdot, w)\big) = \int_D f(z)\overline{K_D(z, w)}\, dV_z$$

for all $f \in H(D)$. This function $K_D(z, w)$ is called the Bergman kernel function for D.

The following lemma is contained in [4].

Lemma 1. *If Ω is a bounded convex domain in $\mathbb{C}^n$ and $0 < R \leqslant 1$, then*

(1) *for any points $p_0 \in \partial\Omega$ and $p_1 \in \Omega$ there exist positive constants C and δ_0 such that the Bergman kernel function K_Ω satisfies the inequality*

$$K_\Omega(p_\delta, p_\delta) > CK_\Omega(p_{\delta/R}, p_{\delta/R})$$

for any $\delta \in (0, \delta_0)$, where $p_\delta := p_0 + \delta(p_1 - p_0)/|p_1 - p_0|$;

(2) *if $0 \in \partial\Omega$ then the scaled domain $R\Omega$ is a thick subdomain of Ω.*

Part (1) of the lemma is identical with [4, Lemma 4.1, part (1)], and we omit the proof. The proof of the second part of the lemma is contained in [4, proof of the implication (1) $\Rightarrow$ (2) in Theorem 1.1]. We recall the proof for the convenience of the reader.