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J. Math. Anal. Appl. 302 (2005) 484–501
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Chordal Loewner families and univalent Cauchy transforms

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Received 16 October 2003

Available online 2 September 2004

Submitted by S. Ruscheweyh

Abstract

We study chordal Loewner families in the upper half-plane and show that they have a parametric representation. We show one, that to every chordal Loewner family there corresponds a unique measurable family of probability measures on the real line, and two, that to every measurable family of probability measures on the real line there corresponds a unique chordal Loewner family. In both cases the correspondence is being given by solving the chordal Loewner equation. We use this to show that any probability measure on the real line with finite variance and mean zero has univalent Cauchy transform if and only if it belongs to some chordal Loewner family. If the probability measure has compact support we give two further necessary and sufficient conditions for the univalence of the Cauchy transform, the first in terms of the transfinite diameter of the complement of the image domain of the reciprocal Cauchy transform, and the second in terms of moment inequalities corresponding to the Grunsky inequalities.

1. Introduction

In this paper we discuss chordal Loewner families, the chordal Loewner equation, and probability measures on the real line whose reciprocal Cauchy transform is univalent in the upper half-plane.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.08.017

Reciprocal Cauchy transforms of probability measures on the real line play an important role in describing the sum of two noncommutative random variables, namely for the free additive convolution developed by Voiculescu [14], and the monotonic convolution developed by Muraki [11].

In [13], Schramm introduced a family of random compact sets, growing in a domain of the complex plane. He showed that any random, growing, and compact set that satisfies a certain Markovian-type and conformal invariance property belongs to this family, and that it can be generated by solving Loewner's equation driven by a Brownian motion on the boundary of the domain. This family is now known as stochastic (or Schramm–) Loewner evolution (SLE). Its discovery soon lead to rigorous proofs of various conjectures of conformal field theory about the behavior of certain statistical mechanical systems at criticality, see [15] and references therein.

In [2], we noted that a solution of the (chordal) Loewner equation at a fixed time is the reciprocal Cauchy transform of some probability measure on the real line. Since any solution of Loewner's equation takes values in the set of univalent functions this raised the question of what characterizes probability measures whose reciprocal Cauchy transform is univalent in the upper half-plane. In particular, does any such measure arise by solving a suitable Loewner equation, and if so, what kind of driving functions need to be considered?

To begin to treat this question we found it necessary to study the chordal Loewner equation beyond the cases we found in the literature. These being either to narrow for our purposes, such as the case of compact complement for SLE, [9], or to general, as in [5], where, at least to our knowledge, no consistent normalization and thus parametrization of chordal Loewner families with a complete correspondence with driving functions is possible. On the other hand, for the (radial) Loewner equation on the unit disk $\mathbb{D}$ there exists just such a treatment, given in [12]. In that case it is convenient to normalize a univalent function f on $\mathbb{D}$ by $f(0) = 0$ and $f'(0) > 0$. (Radial) Loewner families, i.e., maximal subordination chains of such functions, are then parametrized by the derivative at $z = 0$ and one can show that there is a one-to-one correspondence between (radial) Loewner families and so-called Herglotz families, the correspondence being given by solving the (radial) Loewner equation.

In the chordal case in the upper half-plane we have to deal with compactness questions that do not arise in the (radial) disk case. A suitable class of univalent functions to consider are those f that map the upper half-plane into the upper half plane and satisfy

$$\left|f(z) - z\right| \leqslant \frac{C}{\Im(z)}$$

for some $C > 0$ for all z in the upper half-plane. Such functions are in fact reciprocal Cauchy transforms of probability measures on the real line with finite variance and mean zero. We show that the least constant C in the above inequality serves as a parameter for chordal Loewner families and that chordal Loewner families are in one-to-one correspondence with measurable families of probability measures on the real line, the correspondence being given by solving the chordal Loewner equation. The structure of our proof of these results is identical to the structure of the proof of the analogous result in the radial case in [12]. However, the basic tools and inequalities used at the various steps in the argument are very different. We give a detailed proof in Sections 4 and 5, taking up the