Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 302 (2005) 543–556

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

# A note on nonlinear stability of plane parallel shear flows ☆

R. Kaiser [a], G. Mulone [b,*]

[a] *Fakultät für Mathematik und Physik, Universität Bayreuth, D-95440 Bayreuth, Germany*
[b] *Dipartimento di Matematica e Informatica, Città Universitaria, 95125 Catania, Italy*

Received 6 August 2004

Submitted by B. Straughan

**Abstract**

We present a generalized energy functional $\mathcal{E}$ for plane parallel shear flows which provides conditional nonlinear stability for Reynolds numbers Re below some value $\mathrm{Re}_{\mathcal{E}}$ depending on the shear profile. In the case of the experimentally important profiles, viz. combinations of laminar Couette and Poiseuille flow, $\mathrm{Re}_{\mathcal{E}}$ is shown to be at least 174.


*Keywords:* Nonlinear stability; Parallel shear flows; Energy functional; Lyapunov method

## 1. Introduction

Although plane parallel shear flows of viscous incompressible fluids belong to the simplest hydrodynamical systems the stability of the basic flow is up to the present insufficiently understood. It is of particular interest to determine that value of the Reynolds number at which the onset of instability occurs. Most interesting are those shear flows

☆ This work is partially supported by the Italian Ministry for University and Scientific Research, PRIN "Problemi matematici non lineari di propagazione e stabilità nei modelli del continuo," by "Gruppo Nazionale della Fisica Matematica" of the "Istituto Nazionale di Alta Matematica," and by the University of Catania.
* Corresponding author.
*E-mail address:* mulone@dmi.unict.it (G. Mulone).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2004.08.025

which are of experimental relevance. These are the wall and pressure driven flows in plane parallel channels (Couette and Poiseuille flow, respectively) and linear combinations thereof.

The classical methods which yield rigorous stability results, are the energy method and the method of linearized stability. The former method yields global asymptotic stability for Reynolds numbers Re below some value $\mathrm{Re}_\mathrm{E}$ which is of the order $10^2$ for the above mentioned flows [1,8]. The second method yields a critical value $\mathrm{Re}_\mathrm{c}$ below which the system is conditionally stable and above it is unstable. For Poiseuille flow there is $\mathrm{Re}_\mathrm{c}$ of order $10^4$, as numerically has been found (cf. [4]), whereas for Couette flow there is even $\mathrm{Re}_\mathrm{c} = \infty$ [16]. Experimentally, the onset of instability is observed for Reynolds numbers of the order $10^3$ (cf. [3,7]), i.e., none of the classical methods describes the stability behavior satisfactorily.

A third method which has successfully been applied to a couple of hydrodynamic stability problems uses generalized energy functionals $\mathcal{E}$ which are better adjusted to the specific problems under consideration [6,9,17]. This method of generalized energy functionals or Lyapunov direct method [5] provides in general conditional stability for Reynolds numbers below some value $\mathrm{Re}_\mathcal{E}$ together with explicit stability balls in the $\mathcal{E}^{1/2}$-norm. It has already been applied to plane parallel shear flows, however, under the assumption of stress-free boundary conditions for the perturbations [15]. Rigid boundary conditions, which are more appropriate, proved so far as a serious problem for the application of this method [11]. Only recently, using a refined calculus inequality, this problem could be resolved, and conditional nonlinear stability has been proved in the Couette flow case for Reynolds numbers below $\mathrm{Re}_\mathcal{E} = 177$ [12].

Here, conditional nonlinear stability is proved for arbitrary plane parallel shear flows up to some value $\mathrm{Re}_\mathcal{E}$ which depends on the shear profile. The corresponding functional $\mathcal{E}$ is simpler than that used in [12]. As a consequence $\mathrm{Re}_\mathcal{E}$ turns out to be $\mathrm{Re}_\mathrm{E}^x$, the ordinary energy stability limit for perturbations which do not vary in the spanwise direction. In the case of the experimentally important profiles, viz. linear combinations of Couette and Poiseuille flow, this number is at least 174, the value for pure Poiseuille flow. For Couette flow it coincides with the value obtained in [12].

## 2. Preliminaries

The appropriate geometrical setting for plane parallel shear flows is an infinite layer $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$ of thickness 1 with horizontal coordinates $x, y$ and vertical coordinate $z$. Plane parallel shear flows are then characterized by the functional form

$$\mathbf{U}_0 = \mathbf{U}_0(z) = \mathrm{Re}\begin{pmatrix} f(z) \\ 0 \\ 0 \end{pmatrix}. \tag{2.1}$$

The function $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ is assumed to be sufficiently smooth and is called the shear profile. For Couette flow there is $f(z) = -z$ and for Poiseuille flow $f(z) = 1 - 4z^2$. $\mathrm{Re} > 0$