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H_w^p boundedness of Riesz transforms

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Abstract

We show the H_w^p, $0 < p \leqslant 1$, boundedness of Riesz transforms by atomic decomposition and molecular characterization.

Keywords: Atomic decomposition; Hardy spaces; Molecular characterization; Riesz transforms

Let R_j, $j = 1, 2, \dots, n$, denote the *Riesz transforms* in $\mathbb{R}^n$ defined by

$$R_j f(x) = \text{p.v.}\, K_j * f(x), \quad \text{where } K_j(x) = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)\frac{x_j}{|x|^{n+1}}.$$

Use "ˆ" and "ˇ" to denote the Fourier transform and its inverse transform, respectively. Then

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

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Using the system of conjugate harmonic functions, Stein and Weiss [11] showed the H^1 boundedness of the Riesz transforms. By much the same methods as [11], the Riesz transforms have extensions to H^p boundedness for $(n-1)/n < p \leqslant 1$. Later on Fefferman and Stein [3, Theorem 12] extended the result to $0 < p \leqslant 1$ by checking kernels K_j being of class C^∞ away from the origin, and

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} K_j(x)\right| \leqslant B|x|^{-n-|\alpha|} \quad \text{for all multi-indices } \alpha.$$

Theorem A. *The Riesz transforms are bounded on* $H^p(\mathbb{R}^n)$, $0 < p \leqslant 1$.

Another simple proof of Theorem A is to apply the following Hörmander $H^p(\mathbb{R}^n)$ multiplier theorem given by Calderón and Torchinsky [1].

Theorem B. *Let* $0 < p \leqslant 1$ *and* $m \in C^k(\mathbb{R}^n \setminus \{0\})$, $k > n(1/p - 1/2)$. *If* $m \in L^\infty(\mathbb{R}^n)$ *and*

$$\sup_{R>0} R^{2|\alpha|-n} \int\limits_{R<|\alpha|\leqslant 2R} \left|\left(\frac{\partial}{\partial x}\right)^{\alpha} m(x)\right|^2 dx < \infty \quad \textit{for all } |\alpha| \leqslant k,$$

then the multiplier operator mapping f *into* $(m\hat{f})\check{}$ *is bounded on* $H^p(\mathbb{R}^n)$.

If we set $m_j(x) = -ix_j/|x|$, then

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} m_j(x)\right| \leqslant B|x|^{-|\alpha|} \quad \text{for all multi-indices } \alpha.$$

Hence m_j satisfies the assumptions of Theorem B, and we get the H^p, $0 < p \leqslant 1$, boundedness of the mapping $f \mapsto R_j f$.

We are also able to apply Coifman's atomic decomposition [2,8] and Taibleson–Weiss' molecular characterization [12] to prove Theorem A (cf. [10, Chapter 4, Section 4]). Moreover, a more general result can be found in [4, p. 67] and [13, p. 86].

In [9] Lee and Lin established the weighted molecular theory and combined with Garcia-Cuerva's atomic decomposition [5] for weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ to extend Theorem A to the boundedness on $H^p_w(\mathbb{R}^n)$, $n/(n+1) < p \leqslant 1$. In this article, we will extend Theorem A to $H^p_w(\mathbb{R}^n)$ boundedness for $0 < p \leqslant 1$.

In this article a weight means the A_p weight. Let us recall the definition and properties of A_p. For $1 < p < \infty$, a locally integrable nonnegative function w on $\mathbb{R}^n$ is said to belong to A_p if there exists $C > 0$ such that, for every n dimensional cube $I \subset \mathbb{R}^n$,

$$\left(\frac{1}{|I|}\int\limits_I w(x)\,dx\right)\left(\frac{1}{|I|}\int\limits_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant C,$$

where $|I|$ denotes its Lebesgue measure. For the case $p = 1$, $w \in A_1$ if there exists $C > 0$ such that, for every cube $I \subset \mathbb{R}^n$,

$$\frac{1}{|I|}\int\limits_I w(x)\,dx \leqslant C \operatorname*{ess\,inf}_{x\in I} w(x).$$