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On discretizations of invariant foliations over inertial manifolds

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Abstract

Invariant foliations over inertial manifolds of partial differential equations under numerical discretizations are studied. It is proved that the numerical method considered as a discrete dynamical system has C^1-close invariant foliations. The rate of the C^1-convergence is estimated as well.

Keywords: Invariant foliation; Inertial manifold; Discretization; C^1-estimate

1. Introduction

In the study of the dynamics near an equilibrium point, a periodic orbit or a more general invariant set, invariant manifolds and foliations serve as a convenient coordinate system to describe the qualitative behavior of the local flow, see, e.g., [2,5,13,14]. In many cases they are useful tools for technical estimates which facilitate the study of the local bifurcation diagram. Several other important concepts in dynamical systems are closely related to invariant manifolds and foliations. For instance, in [12] detailed results have been given on invariant foliations and their applications to linearizations for finite dimensional systems,

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while in [1] invariant foliations have been used to study partial linearization for noninvertible mappings near fixed points. Recently, in [15] it has been shown that the existence of certain invariant foliations is equivalent to the linearizability for hyperbolic fixed points. Invariant foliations around normally hyperbolic invariant manifolds for finite dimensional systems have been obtained in [9], see also [16]. An extension to infinite dimensional systems has been given by [3].

In this paper we consider a class of evolution partial differential equations that are known to possess an inertial manifold (i.e., a smooth finite dimensional exponentially attracting invariant manifold). Invariant foliations over inertial manifolds have been constructed in [4] (for a similar result we refer to [6]) via an abstract result on the existence and smoothness of the solutions to Lyapunov–Perron equations. The aim of the present paper is to study the behavior of these foliations under numerical approximations. Although such an investigation can be carried out within the framework of perturbation theory, we note that neither [3] nor [4] dealt with the persistence of invariant foliations under perturbations. On the other hand the numerical approximations of inertial manifolds are well studied in the literature, see, e.g., [7,10,11] and references therein. In [10] it has been shown that certain numerical approximations (which might originate from fully discrete or semidiscrete approximations) have approximating inertial manifolds. We extend the results of [10] and show that the invariant foliations over the approximating inertial manifolds converge in the C^1-norm to the original invariant foliation. Moreover, under additional hypotheses we study the rate of the C^1-convergence as well. (Higher order estimates on the rate of the C^0-convergence of the inertial manifolds and their approximations under higher order Runge–Kutta time-discretizations have been given by [7].) Results on the numerics of invariant foliations near fixed points of finite dimensional systems have been given by [8].

2. Preliminaries

First we recall the abstract result of [4] concerning the existence and smoothness of invariant manifolds and foliations for C^1 mappings. For the reader's convenience most of our notations follows [4] as well since we place our problem into the general framework presented there.

Let $(X, \|\cdot\|)$ be a Banach space. We consider a map G of the form

$$G(u) = Lu + N(u),$$

where $L \in L(X, X)$ is a bounded linear operator and $N : X \to X$ is a C^1-map.

From now on we assume that the map G has the following properties.

(G1) There are subspaces X_i, $i = 1, 2$, of X and continuous projections $P_i : X \to X_i$, such that $P_1 + P_2 = I$, $X = X_1 \oplus X_2$, L leaves X_1 and X_2 invariant and L commutes with P_i, $i = 1, 2$. Denoting by $L_i : X_i \to X_i$ the restriction of L on X_i, L_1 has bounded inverse and there exist constants $\alpha_1 > \alpha_2 \geqslant 0$ and $C_1, C_2 \geqslant 1$ such that

$$\left\| L_1^{-k} P_1 \right\| \leqslant C_1 \alpha_1^{-k}, \quad \left\| L_2^k P_2 \right\| \leqslant C_2 \alpha_2^k, \quad k \geqslant 0.$$