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Best constants for tensor products of Bernstein type operators ☆

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Abstract

For the tensor product of k copies of the same one-dimensional Bernstein-type operator L, we consider the problem of finding the best constant in preservation of the usual modulus of continuity for the l_p-norm on $\mathbb{R}^k$. Two main results are obtained: the first one gives both necessary and sufficient conditions in order that $1 + k^{1-1/p}$ is the best uniform constant for a single operator; the second one gives sufficient conditions in order that $1 + k^{1-1/p}$ is the best uniform constant for a family of operators. The general results are applied to several classical families of operators usually considered in approximation theory. Throughout the paper, probabilistic concepts and methods play an important role.

Keywords: Best constant; Bernstein-type operators; Tensor product; Gamma operators; Beta operators; Baskakov operators; Modulus of continuity; Gamma distribution; Beta distribution; Negative binomial distribution; Poisson process; Gamma process

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1. Introduction and main results

Let Δ_k be a (nonempty) convex subset of $\mathbb{R}^k$, and let $L^{\langle k\rangle}$ be a positive linear operator acting on a set $\mathcal{L}^{\langle k\rangle}$ of real functions on Δ_k, which assigns a real function $L^{\langle k\rangle} f$ on Δ_k to each $f \in \mathcal{L}^{\langle k\rangle}$. The problem of global smoothness preservation can be described as the problem of obtaining estimates of the form

$$\omega_p(L^{\langle k\rangle} f;\delta) \leqslant C(\delta)\omega_p(f;\delta), \quad \delta > 0,\ f \in \mathcal{L}^{\langle k\rangle}, \tag{1}$$

where $C(\delta)$ is a positive constant not depending upon f, and

$$\omega_p(f;\delta) := \sup\bigl\{\bigl|f(\mathbf{x}) - f(\mathbf{y})\bigr|\colon \mathbf{x},\mathbf{y} \in \Delta_k,\ \|\mathbf{x}-\mathbf{y}\|_p \leqslant \delta\bigr\}$$

is the usual modulus of continuity for the l_p-norm on $\mathbb{R}^k$ ($p \in [1,\infty]$). In particular, it is interesting to determine the value of the best possible constant on the right-hand side in (1). Provided that $L^{\langle k\rangle} f$ is constant whenever f is, such a best constant is obviously given by

$$C_p^{\langle k\rangle}(\delta) := \sup_{f\in\mathcal{L}_*^{\langle k\rangle}} \frac{\omega_p(L^{\langle k\rangle} f;\delta)}{\omega_p(f;\delta)}, \quad \delta > 0, \tag{2}$$

where

$$\mathcal{L}_*^{\langle k\rangle} := \bigl\{f \in \mathcal{L}^{\langle k\rangle}\colon 0 < \omega_p(f;1) < \infty\bigr\}.$$

Problems of this kind have been discussed in several works by using different approaches (see, for instance, [1–10,12] and references therein). The probabilistic approach developed in [1,2,6–9,12] has proved to be suitable and fruitful when dealing with operators of probabilistic type (also called Bernstein-type operators), that is, operators allowing for a representation of the form

$$L^{\langle k\rangle} f(\mathbf{x}) = Ef\bigl(\xi^{\langle k\rangle}(\mathbf{x})\bigr), \quad \mathbf{x} \in \Delta_k,\ f \in \mathcal{L}^{\langle k\rangle}, \tag{3}$$

where E denotes mathematical expectation, $\{\xi^{\langle k\rangle}(\mathbf{x})\colon \mathbf{x} \in \Delta_k\}$ is a stochastic process taking values in Δ_k, and $\mathcal{L}^{\langle k\rangle}$ is the set of all real functions on Δ_k for which the right-hand side in (3) makes sense.

In the present paper, we consider k-dimensional operators which are tensor products of k copies of the same one-dimensional Bernstein-type operator. More precisely, the setting is the following.

Let I be the interval $[0,1]$ or $[0,\infty)$, and let L be the Bernstein-type operator over I given by

$$Lf(x) := Ef\bigl(\xi(x)\bigr), \quad x \in I,\ f \in \mathcal{L},$$

where the I-valued stochastic process $\{\xi(x)\colon x \in I\}$ is assumed to be integrable (this always holds when $I = [0,1]$). It is easy to see that the domain $\mathcal{L}$ contains all the real (measurable, when needed) functions on I such that $\omega(f;1) < \infty$.

The tensor product $L^{\langle k\rangle} := L \otimes \cdots \otimes L$ of k copies of L is the k-dimensional operator over $\Delta_k := I^k$ given by (3), where

$$\xi^{\langle k\rangle}(\mathbf{x}) := \bigl(\xi_1(x_1),\ldots,\xi_k(x_k)\bigr), \quad \mathbf{x} := (x_1,\ldots,x_k) \in I^k, \tag{4}$$