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Eliminations in Weyl algebras and identities ☆

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Abstract

In this article, we study the eliminations in noncommutative operator algebras and modify the Zeilberger algorithm, so that terminating hypergeometric identities, q-proper-hypergeometric identities as well as identities with the integral sign can be proven automatically. Besides, based on the Wu method, we also give an algorithm for proving multivariate hypergeometric identities.

Keywords: Computer proofs; Elimination; Weyl algebra; Combinatorial identity; Zeilberger algorithm; Wu method

1. Introduction

In [7], Zeilberger presented an approach to special functions identities including all terminating hypergeometric identities, besides providing a general mathematical framework for the proof machinery. This leads to mathematically and algorithmically challenging questions such as elimination in noncommutative operator algebras (Weyl algebras).

The intention of the Zeilberger algorithm is to find the recurrence relations satisfied by both sides of the identity. What it makes use of is Sylvester's classical dialytic elimina-

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tion, which requires a lot of memory. Some identities, such as Dixon's identity, cannot be verified successfully.

Taking into account of this problem, we develop eliminations in noncommutative algebras of operators, i.e., shift operator, dilation operator, and differential operator. And we modify the Zeilberger algorithm by the eliminations. With the modified Zeilberger algorithm, terminating hypergeometric identities, q-proper-hypergeometric identities as well as identities with the integral sign, can be proven automatically. Besides, with the Wu method, a set of mechanical approaches established by Wu Wentsun [4,5], we present an automatic approach to proving terminating hypergeometric identities of several variables.

This article is organized as follows. Elimination under the shift operators will be studied in Section 2. In this section, we will generalize the Euclidean algorithm to the noncommutative context of the Weyl algebra, and then give the main theorem of this article, that is, given two polynomials $A, B \in C\langle N, K, n\rangle$, there always exist polynomials $u, v \in C\langle N, K, n\rangle$ such that $uA = vB$, where N, K are the shift operators according to n, k, respectively. Finally, we will show how to eliminate indeterminate k from two polynomials P, Q which belong to $C\langle N, K, n, k\rangle$. In Section 3, we will present the modified Zeilberger algorithm. A Maple program based on it will also be given there. Basic idea for verifying multivariate identities as well as the corresponding algorithm will be introduced in Section 4. And at last, in Section 5, we will give the final conclusions on dilation operator and differential operator, which lead to the algorithms of proving q-proper-hypergeometric identities and identities with the integral sign, respectively.

2. Elimination in Weyl algebra

Let us define the shift operators N and K by $Nf(n,k) := f(n+1,k)$, $Kf(n,k) := f(n,k+1)$. A partial recurrence operator with polynomial coefficients is any polynomial in the four indeterminates N, K, n, k. They satisfy the commutation relations: $Nn = (n+1)N$, $Kk = (k+1)K$, $NK = KN$, $Nk = kN$, $nK = Kn$, $nk = kn$. And $C\langle N, K, n, k\rangle$ is the noncommutative algebra generated by the indeterminates N, K, n, k.

Given two polynomials $A, B \in C\langle N, K, n\rangle$, A, B could be considered as polynomials in n:

$$A = a_i n^i + a_{i-1} n^{i-1} + \cdots + a_0, \qquad a_i \neq 0,$$
$$B = b_j n^j + b_{j-1} n^{j-1} + \cdots + b_0, \qquad b_j \neq 0,$$

where $a_i, b_j \in C\langle N, K\rangle$, $a_i b_j = b_j a_i$. The following theorem can be achieved.

Theorem 1. *For any given polynomials $A, B \in C\langle N, K, n\rangle$, there always exist polynomials $u, v, r \in C\langle N, K, n\rangle$, such that $uA = vB + r$, where $\deg_n r < \deg_n B$. r is called the remainder of A to B and denoted by* $\mathrm{Rem}(A/B)$.

Proof. If $i < j$, $\deg_n A < \deg_n B$, we can choose $u = 1$, $v = 0$, $r = A$.

If $i = j$, $\deg_n A = \deg_n B$, since $a_i b_j = b_j a_i$, we can choose $u = b_j$, $v = a_i$ so that $r = b_j A - a_i B$ and $\deg_n r < \deg_n B$.