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Dipaths and dihomotopies in a cubical complex

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Abstract

In the geometric realization of a cubical complex without degeneracies, a □-set, dipaths and dihomotopies may not be combinatorial, i.e., not geometric realizations of combinatorial dipaths and equivalences. When we want to use geometric/topological tools to classify dipaths on the 1-skeleton, combinatorial dipaths, up to dihomotopy, and in particular up to combinatorial dihomotopy, we need that all dipaths are in fact dihomotopic to a combinatorial dipath. And moreover that two combinatorial dipaths which are dihomotopic are then combinatorially dihomotopic. We prove that any dipath from a vertex to a vertex is dihomotopic to a combinatorial dipath, in a non-selfintersecting □-set. And that two combinatorial dipaths which are dihomotopic through a non-combinatorial dihomotopy are in fact combinatorially dihomotopic, in a geometric □-set. Moreover, we prove that in a geometric □-set, the d-homotopy introduced in [M. Grandis, Directed homotopy theory, I, Cah. Topol. Géom. Différ. Catég. 44 (4) (2003) 281–316] coincides with the dihomotopy in [L. Fajstrup, E. Goubault, M. Raussen, Algebraic topology and concurrency, Theoret. Comput. Sci., in press; also technical report, Aalborg University, 1999].

Keywords: Ditopology; Combinatorial ditopology; Dihomotopy; Automata; Cubical complex; □-set

1. Introduction

The relatively new subject of directed topology and geometry, ditopology, has a combinatorial/algebraic as well as a geometric/topological approach. The subject originates in computer science, where V. Pratt in [10] introduces higher dimensional automata, HDA,

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as a model for concurrency. This is both an algebraic and a geometric model: The HDA is a cubical complex without degeneracies, a $\square$-set, which may be geometrically realized [5] as a locally partially ordered space, or be treated algebraically.

On the combinatorial/algebraic side, R. van Glabbeek, in a still unpublished note [13] defined a notion of bisimulation, an equivalence relation on HDAs. This is an equivalence on generalized discrete paths, and the results presented here will be necessary for setting up the connection to geometric notions of equivalence of directed continuous paths.

Another computer scientific application is to PV-models, [2], a "toy" language, which gives the overall structure of the underlying programs in terms of their interaction through shared resources, and which on the geometric side corresponds to products of directed graphs (possibly with loops) from which certain subspaces have been removed—the forbidden area.

Example 1.1. Suppose two processors T_1 and T_2 share resources A and B. Suppose moreover, that these resources can serve only one process at a time. In Dijkstra's PV-model a processor locks the resource A while using it (denote this action P_A) and releases it, denoted V_A, when it has finished using it.

The concurrent program T_1: $P_A P_B V_B V_A$ and T_2: $P_B P_A V_A V_B$ is geometrically represented by the "Swiss Flag", see Fig. 1, where a joint execution of the programs is represented as a continuous path from the lower left corner to the upper right corner avoiding the black area. The paths should be increasing in both coordinates, they are dipaths.

Combinatorial models of geometry are certainly well studied, and cubes as building blocks are used in many applications. The contribution in this paper is to keep the (time) directions preserved as required in the applications.

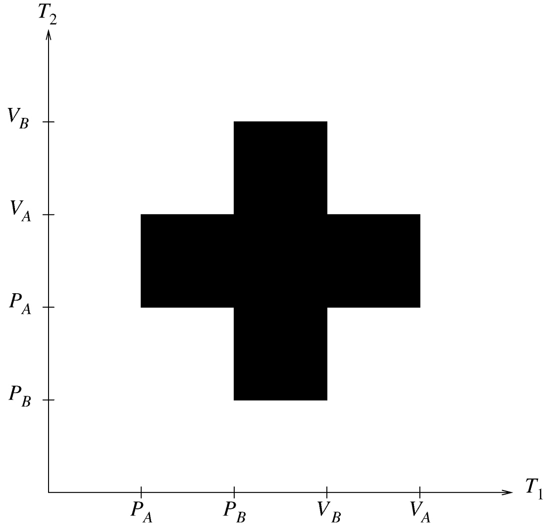

Fig. 1. The Swiss Flag example.