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Markov's transformation of series and the WZ method

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Abstract

In a well forgotten memoir of 1890, Andrei Markov devised a convergence acceleration technique based on a series transformation which is very similar to what is now known as the Wilf–Zeilberger (WZ) method. We review Markov's work, put it in the context of modern computer-aided WZ machinery, and speculate about possible reasons of the memoir being shelved for so long.

Keywords: Wilf–Zeilberger method; A.A. Markov, Sr.; Series transformation; Convergence acceleration; Hypergeometric series; Basic hypergeometric series; Double series; Discrete Green formula; Apéry constant

1. Introduction

By this publication we aim to resurface the memoir [17] by the Russian mathematician Andrei Andreevich Markov (1856–1922), who is best known as the inventor of Markov's chains in probability theory. However, by the time Markov began his studies in probability, he was a distinguished analyst and a member of the (Russian) Emperor's Academy of Sciences.

Why would the old paper be worth attention of today's mathematical community? All of the sudden, it appears very relevant in the context of a powerful technique of series

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transformation known as the Wilf–Zeilberger (WZ) method, and just as relevant in the context of recent sport about faster and faster evaluation of the constant

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots,$$

called the Apéry constant after R. Apéry proved its irrationality in 1978 [1]. (Not too far away is an actively pursued challenge—irrationality of further odd zeta values, cf. [35] and references therein.)

To appreciate the following results, try (if you never did) to obtain 7 correct decimals of $\zeta(3)$ with a non-programmable calculator!

The formula

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{n} n^3} \tag{1}$$

is often attributed to Apéry, but it wasn't him who first discovered it. The review [26] points out the result [13] reported in 1953, and here is formula (14) from Markov's memoir

$$\sum_{n=0}^{\infty} \frac{1}{(a+n)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n n!^6}{(2n+1)!} \frac{5(n+1)^2 + 6(a-1)(n+1) + 2(a-1)^2}{[a(a+1)\cdots(a+n)]^4}, \tag{2}$$

which is a generalization of (1). The series (1), (2) converge at the geometric rate with ratio $1/4$. A series convergent at the geometric rate with ratio $1/27$,

$$\zeta(3) = \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{56n^2 - 32n + 5}{(2n-1)^2 n^3} \frac{(n!)^3}{(3n)!}, \tag{3}$$

is "automatically" derived in [2], together with formula (1), using the WZ method; see also [31]. Interestingly, Markov has an equivalent of (3) on page 9 of his memoir.

Note for reference that [2] contains an even faster convergent representation for $\zeta(3)$ with ratio $2^2/4^4 = 1/64$. A series of a non-hypergeometric type, convergent at the geometric rate with ratio $e^{-2\pi} \approx 1/535$ is essentially due to Ramanujan [6, p. 30, (59)]. And the largest number of decimals in $\zeta(3)$, currently 520 000, to our knowledge, was obtained by means of the nice formula derived in [3]

$$\zeta(3) = \sum_{n=0}^{\infty} (-1)^n \frac{n!^{10}\,(205n^2 + 250n + 77)}{64\,(2n+1)!^5}.$$

The ratio of convergence here is 2^{-10}.

These highly nontrivial results have been obtained by the same method, which is deceitfully simple in an abstract form. It can be viewed either as a generalization of the