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Numerical integration over smooth surfaces in $\mathbb{R}^3$ via class $\mathscr{S}_m$ variable transformations. Part I: Smooth integrands

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Abstract

Class $\mathscr{S}_m$ variable transformations with integer m for finite-range integrals were introduced by the author about a decade ago. These transformations "periodize" the integrand functions in a way that enables the trapezoidal rule to achieve very high accuracy, especially with even m. In a recent work by the author, these transformations were extended to *arbitrary* real m, and their role in improving the convergence of the trapezoidal rule for different classes of integrands was studied in detail. It was shown that, with m chosen appropriately, exceptionally high accuracy can be achieved by the trapezoidal rule. For example, if the integrand function is smooth on the interval of integration including the endpoints, and vanishes at the endpoints, then excellent results are obtained by taking $2m$ to be an odd integer. In the present work, we consider the use of these transformations in the computation of integrals on surfaces of simply connected bounded domains in $\mathbb{R}^3$, in conjunction with the product trapezoidal rule. We assume these surfaces are smooth and homeomorphic to the surface of the unit sphere, and we treat the cases in which the integrands are smooth. We propose two approaches, one in which the product trapezoidal rule is applied with the integrand as is, and another, in which the integrand is preprocessed before the rule is applied. We give thorough analyses of the errors incurred in both approaches, which show that surprisingly

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high accuracies can be achieved with suitable values of m. We also illustrate the theoretical results with numerical examples.

Keywords: Numerical integration; Integration over smooth surfaces in $\mathbb{R}^3$; Product trapezoidal rule; Variable transformations; $\sin^m$-transformation; Euler–Maclaurin expansions; Asymptotic expansions

1. Introduction

In this work, we present a new approach to the numerical evaluation of integrals over smooth surfaces in three dimensions. We treat integrals of the form

$$I[f] = \iint_S f(Q)\,\mathrm{d}A_S, \tag{1.1}$$

where S is the surface of an arbitrary bounded and simply connected domain in $\mathbb{R}^3$ and $\mathrm{d}A_S$ is the associated area element. We assume that S is infinitely smooth and homeomorphic to the surface of the unit sphere, which we shall denote by U throughout. We also assume that the transformation from U to S is one-to-one and infinitely differentiable and that it has a nonsingular Jacobian matrix.

The integrand functions $f(Q)$ we consider are smooth over S. (In another work [19], we treat the cases in which the integrand functions have point singularities of the single-layer and double-layer types over S.)

Such integrals, with smooth or singular $f(Q)$, arise in boundary integral equation formulations of partial differential equations in continuum problems. For a review of this subject, see Atkinson [2] and [3, Chapter 5].

Here are the steps of the basic method of integration we present in this work:

(i) Using the mapping of U, the surface of the unit sphere, to S, express $I[f]$ as an integral over U.
(ii) Express the (transformed) integral over U in terms of the standard spherical coordinates θ and ϕ, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. The resulting integral can be written in the form $I[f] = \int_0^{\pi}\left[\int_0^{2\pi} F(\theta,\phi)\,\mathrm{d}\phi\right]\mathrm{d}\theta$.
(iii) Transform θ by a variable transformation $\theta = \Psi(t) \equiv \pi\psi(t)$, $0 \leqslant t \leqslant 1$, where $\psi(t)$ is in the extended class $\mathscr{S}_m$ of Sidi [22]. The result of this is $I[f] = \int_0^1\left[\int_0^{2\pi} F(\Psi(t),\phi)\,\mathrm{d}\phi\right]\Psi'(t)\,\mathrm{d}t$.
(iv) Approximate the final integral in the variables t and ϕ by the product trapezoidal rule.