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On the system of high order rational difference equations $x_n = \frac{a}{y_{n-p}}$, $y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}$

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Abstract

In this paper, we study the behavior of positive solutions of the system of high order rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}, \quad n = 1, 2, \ldots,$$

where p and q are positive integers with $p \leqslant q$, and a and b are positive constants.

Keywords: Difference equation; Periodicity; Monotonicity; Recurrence relation

1. Introduction

In recent years, the behavior of positive solutions of various systems of rational difference equations has been one of the main topics in the theory of

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difference equations [1–5]. In particular, Cinar [1] proved that all positive solutions of the system of difference equations

$$x_n = \frac{1}{y_{n-1}}, \quad y_n = \frac{y_{n-1}}{x_{n-2}y_{n-2}}, \quad n = 1, 2, \ldots \tag{1.1}$$

are period four.

In this paper, we study the system of high order difference equations

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}, \quad n = 1, 2, \ldots, \tag{1.2}$$

where p and q are positive integers with $p \leqslant q$, and a and b are positive constants. When $p = 1$, $q = 2$, and $a = b = 1$, Eq. (1.2) reduces to Eq. (1.1).

2. Main results

We first examine the periodicity of positive solutions of Eq. (1.2).

Theorem 2.1. *If $a = b$, and q is divisible by p, then every positive solution of Eq. (1.2) is period $2q$.*

Proof. Assume $q = rp$, where r is a positive integer. Consider an arbitrary positive solution $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ of Eq. (1.2). For each $n \geqslant q + 1$, replacing $x_{n-q} = \frac{a}{y_{n-(p+q)}}$ into $y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}$ and multiplying both sides by y_{n-q}, we get the recurrence relation

$$y_n y_{n-q} = y_{n-p} y_{n-(p+q)}. \tag{2.1}$$

It follows by repeatedly applying this recurrence relation that

$$y_n y_{n-q} = y_{n-p} y_{n-(p+q)} = y_{n-2p} y_{n-(2p+q)} = \cdots = y_{n-rp} y_{n-(rp+q)} = y_{n-q} y_{n-2q}.$$

So $y_n = y_{n-2q}$ and hence $x_n = \frac{a}{y_{n-p}} = \frac{a}{y_{n-p-2q}} = x_{n-2q}$. The proof is completed. □

As a direct consequence of this theorem, we again obtain the result in [1] as follows.

Corollary 2.2. *Every positive solution of Eq. (1.1) is period 4.*

We then examine the monotonicity of positive solutions of Eq. (1.2).

Theorem 2.3. *Assume q is divisible by p. Let $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ be an arbitrary positive solution of Eq. (1.2).*