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A numerical solution of Burgers' equation by finite element method constructed on the method of discretization in time

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Abstract

In this paper, the Galerkin finite element method constructed on the method of discretization in time was applied to solve the one-dimensional nonlinear Burgers' equation. The system of nonlinear equations obtained for each time step was solved by using the Newton method. In order to show the efficiency of the presented method, the numerical solutions obtained for various values of viscosity were compared with the exact solutions. It was seen that they were in excellent agreement.

Keywords: Burgers' equation; The method of discretization in time; The Galerkin finite element method

1. Introduction

Burgers' equation which is the one-dimensional nonlinear partial differential equation,

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = \varepsilon\frac{\partial^2 U}{\partial x^2}, \quad a < x < b, \ \ t > 0 \tag{1}$$

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is a mathematical model of both turbulence theory and shock wave theory. In Eq. (1), $\varepsilon > 0$ is small parameter known as the kinematics viscosity of the fluid motion. The distinctive feature of Eq. (1) is that it is the simplest mathematical formulation of the competition between nonlinear advection and the viscous diffusion. The mathematical properties of Eq. (1) had been studied by Cole [1]. Fortunately, Burgers' equation is one of the very few nonlinear partial differential equations, which can be solved exactly owing a transformation for arbitrary initial and boundary conditions. However, it is well known that the exact solution of Burgers' equation can be computed only for restricted values of kinematics viscosity. So, Burgers' equation is taken as a model not only to test the numerical methods but also to obtain the numerical solution of equation for small values of viscosity. Many researchers have used various numerical methods to solve Burgers' equation [2–12]. Recently, it is the popular way to obtain the solution of the problem by the numerical methods constructed on the method of discretization in time [10,12]. In the present work, the finite element method constructed on the method of discretization in time was directly applied to nonlinear Burgers' equation to solve it without using any transformation as Hopf–Cole transformation. The numerical results obtained for various values of viscosity have been compared with the Cole's exact solution. It was seen that the numerical results are very satisfactory.

2. Statement of the problem

Let us consider Burgers' equation (1) with the initial condition

$$U(x,0) = \sin \pi x \quad \text{in } 0 < x < 1 \tag{2}$$

and boundary conditions

$$U(0,t) = 0, \quad U(1,t) = 0, \quad t > 0. \tag{3}$$

The exact solution of Eq. (1) with conditions (2) and (3) was given by Cole [1] as

$$U(x,t) = 2\pi\varepsilon \frac{\sum_{n=1}^{\infty} a_n \mathrm{e}^{-n^2\pi^2\varepsilon t} n \sin n\pi x}{a_0 + \sum_{n=1}^{\infty} a_n \mathrm{e}^{-n^2\pi^2\varepsilon t} \cos n\pi x},$$

where a_0 and a_n ($n = 1, 2, \ldots$) are Fourier coefficients and defined by the following equations

$$a_0 = \int_0^1 \exp[-(2\pi\varepsilon)^{-1}(1 - \cos \pi x)]\,\mathrm{d}x,$$

$$a_n = 2\int_0^1 \exp[-(2\pi\varepsilon)^{-1}(1 - \cos \pi x)] \cos n\pi x\,\mathrm{d}x, \quad n \geqslant 1.$$