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The tanh and the sine–cosine methods for compact and noncompact solutions of the nonlinear Klein–Gordon equation

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Abstract

The nonlinear Klein–Gordon equation is used as a vehicle to employ the tanh method and the sine–cosine method to formally derive a number of travelling wave solutions. The study features a variety of solutions with distinct physical structures. The work shows that one method complements the other, and each method gives solutions of formal properties. The obtained solutions include compactons, solitons, solitary patterns, and periodic solutions.

Keywords: Compactons; Solitons; Periodic solutions; Klein–Gordon equation; Sine–cosine method; The tanh method

1. Introduction

The quasilinear Klein–Gordon equation

$$u_{tt} - au_{xx} + bu - ku^3 = 0, \tag{1}$$

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and the nonlinear Klein–Gordon equation

$$u_{tt} - au_{xx} + f(u) = 0, \tag{2}$$

are used to model many nonlinear phenomena [1], including the propagation of dislocations in crystals and the behavior of elementary particles and the propagation of fluxons in Josephson junctions. The function $f(u)$ takes many forms such as

$$f(u) = \begin{cases} \sin u, \\ \sinh u, \\ e^{u}, \\ e^{u} + e^{-2u}, \\ e^{-u} + e^{-2u}, \end{cases} \tag{3}$$

that characterize the Sine-Gordon equation, sinh-Gordon equation, Liouville equation, Dodd–Bullough–Mikhailov equation (DBM), and the Tzitzeica–Dodd–Bullough (TDB) equation, respectively. Also, $f(u)$ appears as a polynomial such as $f(u) = bu - ku^3$, and $f(u) = bu - ku^n$.

The sine-Gordon and the sinh-Gordon equations gained its importance when it gave *kink* and *antikink* solutions with the collisional behaviors of solitons. A kink is a solution with boundary values 0 and 2π at the left infinity and at the right infinity, respectively [1–4]. However, antikink is a solution with boundary values 0 and -2π at the left infinity and at the right infinity, respectively. Moreover, these two equations appear in many fields such as the propagation of fluxons in Josephson junctions [2] between two superconductors, the motion of rigid pendula attached to a stretched wire, solid state physics, and nonlinear optics.

The last two equations, Dodd–Bullough–Mikhailov equation and the Tzitzeica–Dodd–Bullough equation, appear in problems varying from fluid flow to quantum field theory.

In 1965, Zabusky and Kruskal [4] investigated the interaction of solitary waves and the recurrence of initial states. The term soliton was coined by Zabusky and Kruskal [4], who performed numerical studies of the KdV equation, and found particle like waves which retained their shapes and velocities after collisions. The term soliton is coined to reflect the particle like behavior of the solitary waves under interaction. The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons.

It was formally proved by many that solitons exist due to the balance between the weak nonlinearity and dispersion. Soliton is a localized wave that has an infinite support or a localized wave with exponential wings. The soliton