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Vandermonde systems on Gauss–Lobatto Chebyshev nodes

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Abstract

This paper deals with Vandermonde matrices V_n whose nodes are the Gauss–Lobatto Chebyshev nodes, also called extrema Chebyshev nodes. We give an analytic factorization and explicit formula for the entries of their inverse, and explore its computational issues. We also give asymptotic estimates of the Frobenius norm of both V_n and its inverse and present an explicit formula for the determinant of V_n.

Keywords: Vandermonde matrices; Polynomial interpolation; Conditioning

1. Introduction

Vandermonde matrices defined by $\widetilde{V}_n(i,j) = x_j^{i-1}, i,j = 1,2\ldots,n; x_j \in \mathbb{C}$ are still a topical subject in matrix theory and numerical analysis. The interest arises as they occur in applications, for example in polynomial and exponential interpolation, and because they are ill-conditioned, at least for positive or

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symmetric real nodes [1]. The primal system $\widetilde{V}_n a = b$ represents a moment problem, which arises, for example, when determining the weights for a quadrature rule, while the matrix $V_n = \widetilde{V}_n^{\mathrm{T}}$ involved in the dual system $V_n c = f$ plays an important role in polynomial approximation and interpolation problems [2,3]. The special structure of V_n allows us to use ad hoc algorithms that require $O(n^2)$ elementary operations for solving a Vandermonde system. The most celebrated of them is the one by Björck and Pereyra [4]; these algorithms frequently produce surprisingly accurate solution, even when V_n is ill-conditioned [2]. Bounds or estimates of the norm of both V_n and V_n^{-1} are also interesting, for example to investigate the condition of the polynomial interpolation problem. Answer to these problems have been given first for special configurations of the nodes and recently for general ones [5].

Polynomial interpolation on several set of nodes has received much attention over the past decade [6]. Theoretically, any discretization grid can be used to construct the interpolation polynomial. However, the interpolated solution between discretization points are accurate only if the individual building blocks behave well between points. Lagrangian polynomials with a uniform grid suffer for the effect of the Runge phenomenon: small data near the center of the interval are associated with wild oscillations in the interpolant, on the order 2^n times bigger, near the edges of the interval, [7,8]. The best choice is to use nodes that are clustered near the edges of the interval with an asymptotic density proportional to $(1 - x^2)^{-1/2}$ as $n \to \infty$, [9]. The family of Chebyshev points, obtained by projecting equally spaced points on the unit circle down to the unit interval $[-1, 1]$ have such density properties. The classical Chebyshev grids are [10]:

- *Chebyshev nodes*

$$T_1 = \left\{ x_k = \cos\left[\frac{2k-1}{2n}\pi\right], \quad k = 1, 2, \ldots, n \right\} \tag{1}$$

- *Extended Chebyshev nodes*

$$T_2 = \left\{ x_k = -\frac{\cos\left(\frac{2k-1}{2n}\pi\right)}{\cos\left(\frac{\pi}{2n}\right)}, \quad k = 1, 2, \ldots, n \right\} \tag{2}$$

- *Gauss–Lobatto Chebyshev nodes (extrema)*

$$T_3 = \left\{ x_k = -\cos\left[\frac{k-1}{n-1}\pi\right], \quad k = 1, 2, \ldots, n \right\} \tag{3}$$

In [11] it is proved that interpolation on the Chebyshev polynomial extrema minimizes the diameter of the set of the vectors of the coefficients of all possible polynomials interpolating the perturbed data. Although the set of Gauss–Lobatto Chebyshev nodes failed to be a good approximation to the optimal