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On numerical improvement of the second kind of Gauss–Chebyshev quadrature rules

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Abstract

One of the integration methods is the Second Kind of Gauss–Chebyshev quadrature rule, denoted by:

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f\left(\cos\left(\frac{k\pi}{n+1}\right)\right) + \frac{\pi}{2^{2n+1}(2n)!}f^{(2n)}(\eta), \quad -1<\eta<1.$$

According to Gauss quadrature rules, the precision degree of above formula is the highest, i.e. $2n-1$. Hence, it is not possible to increase the precision degree of Second Kind of Gauss–Chebyshev integration formulas anymore. But, on the other hand, we claim that we can improve the above formula numerically. To do this, we consider the integral bounds as two unknown variables. This causes to numerically be extended the monomial space $f(x)=x^j$ from $j=0,1,\ldots,2n-1$ to $j=0,1,\ldots,2n+1$. This means that we

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have two monomials more than Second Kind Gauss–Chebyshev integration method. In other words, we give an approximate formula as:

$$\int_a^b f(x)\sqrt{1-x^2}\,\mathrm{d}x \simeq \sum_{i=1}^{n} w_i f(x_i),$$

in which a,b and $w_1,w_2,\ldots,w_n$ and $x_1,x_2,\ldots,x_n$ are all unknowns and the formula is almost exact for the monomial basis $f(x) = x^j, j = 0,1,\ldots,2n+1$. Some important examples are finally given to show the excellent superiority of the proposed nodes and coefficients with respect to the Second Kind Gauss–Chebyshev nodes and coefficients. Let us add that in this part we give also some wonderful 2-point, 3-point and 4-point formulas that are respectively comparable with 103-point, 261-point and 108-point formulas of Second Kind Gauss–Chebyshev quadrature rules in average.

Keywords: Second Kind Gauss–Chebyshev formula; Numerical integration methods; Degree of accuracy; The method of undetermined coefficient; The method of solving nonlinear systems

1. Introduction

It is known that the general form of Gauss quadrature rules is indicated as:

$$\int_a^b f(x)\,\mathrm{d}w(x) = \sum_{j=1}^{n} w_j f(x_j) + \sum_{k=1}^{m} v_k f(z_k) + R_{n,m}[f], \tag{1}$$

where the weights $[w_j]_{j=1}^n$, $[v_k]_{k=1}^m$ and nodes $[x_j]_{j=1}^n$ are unknowns and the nodes $[z_k]_{k=1}^m$ are predetermined, w is also a positive measure on $[a,b]$ (see [10–12]).

The residue $R_{n,m}[f]$ is determined (see for instance [13]) by:

$$R_{n,m}[f] = \frac{f^{(2n+m)}(\eta)}{(2n+m)!}\int_a^b \prod_{k=1}^{m}(x-z_k)\left[\prod_{j=1}^{n}(x-x_j)\right]^2 \mathrm{d}w(x), \quad a<\eta<b. \tag{2}$$

By selecting $\mathrm{d}w(x) = \sqrt{1-x^2}\,\mathrm{d}x, a=-1, b=1$ and $m = 0$ we reach the Second Kind Gauss–Chebyshev formula. In other words we have:

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}\,\mathrm{d}x = \sum_{j=1}^{n} w_j f(x_j) + R_n(f), \tag{3}$$

where,

$$R_n(f) = \frac{f^{(2n)}(\eta)}{(2n)!}\int_{-1}^{1}\left[\prod_{i=1}^{m}(x-x_i)\right]^2\sqrt{1-x^2}\,\mathrm{d}x. \tag{4}$$