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All solutions of system of ill-posed operator equations of the first kind

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Abstract

In this paper, we discuss the structure of the solution space of system of operator equations of the first kind $A_i u = f_i$, $i = 1,2,\ldots,n$ in Hilbert spaces. If it has solutions, we give the analytic representation of all its solutions. Final examples show our methods are effective.

Keywords: Ill-posed operator equation; Analytic solutions; Reproducing kernel

1. Introduction

Let H, H_1 be separable Hilbert spaces, and let $A{:}H \to H_1$ be a bounded linear operator. Consider operator equations of the first kind

$$Au = f, \quad u \in H, \quad f \in H_1. \tag{1}$$

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Usually, operator equations of the first kind is ill-posed, so the problem of how to solve Eq. (1) becomes very important. Here, ill-posed means that at least one of the following three conditions cannot be satisfied:

(1) Eq. (1) has solutions for every $f \in H_1$;
(2) Eq. (1) has a unique solution for every $f \in H_1$;
(3) inverse operator A^{-1} is continuous.

Usually, in order to solve Eq. (1), it needs to suppose that A^{-1} is single-valued. In this paper, Only supposing ill-posed Eq. (1) has solutions, we discuss the structure of its solution space and obtained the following results:

(1) If Eq. (1) has solutions, we represent its solution space as $u_0 + N(A)$, where u_0 is the minimal norm solution of Eq. (1), and $N(A)$ is the null space of operator A, namely, $N(A) = \{x \in H | Ax = 0\}$;
(2) a complete orthonormal system of $N(A)$ is given.

For generality, we directly discuss the following system of linear operator Eqs. (I) in Hilbert spaces,

$$\begin{cases} A_1 u & = f_1 \\ A_2 u & = f_2 \\ & \vdots \\ A_n u & = f_n \end{cases} \quad u \in H,\ f_i \in H_1,\ i = 1, 2, \ldots, n, \tag{I}$$

where H, H_1 are separable Hilbert spaces, for every $i = 1,2, \ldots ,n$, $A_i{:}H \to H_1$ is a bounded linear operator.

2. Lemmas

Throughout this paper, we suppose that $A_i{:}H \to H_1$ is a bound linear operator, $N(A_i)$ is the null space of operator A_i, namely, $N(A_i) = \{x \in H | A_i x = 0\}$. For $M \subset H_1$, we denote *span* M by $[M]$ and the closure of M by $\overline{M}$. Suppose $\{r_i\}_{i=1}^{\infty} \subset H$, and we use $\{\bar{r}_i\}_{i=1}^{\infty}$ to denote the system of functions that is obtained from $\{r_i\}_{i=1}^{\infty}$ by Gram–Schmidt process of orthonormalization.

We can easily obtain the following lemma.

Lemma 2.1. *Suppose* $\{0\} \neq \{r_i\}_{i=1}^{\infty}, \{v_i\}_{i=1}^{\infty} \subset H$. *Define* $\{\bar{r}_i\}_{i=1}^{\infty}$ *and* $\{\tilde{v}_i\}_{i=1}^{\infty}$ *as follows*:

$$\bar{r}_1 = \frac{r_1}{\|r_1\|}, \tag{2}$$