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Standing waves for nonlinear Klein–Gordon equations with nonnegative potentials ☆

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Abstract

This paper is concerned with the standing waves for nonlinear Klein–Gordon equations with nonnegative potentials. First, the existence of standing waves associated with the ground states is obtained by using variational calculus as well as a compactness lemma. Next, a series of sharp conditions for global existence of nonlinear Klein–Gordon equations with nonnegative potentials are established in terms of the characteristics of the ground state and the local theory. Then, that how small the initial data are, the global solutions exist is given. Finally, the instability of standing wave is shown by combining those results.

Keywords: Nonlinear Klein–Gordon equation; Nonnegative potential; Ground state; Global existence; Blowup; Instability

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1. Introduction

In the present paper, we consider the nonlinear Klein–Gordon equations with a real valued potential $V(x)$:

$$\phi_{tt} = \Delta\phi - m^2\phi - V(x)\phi + a|\phi|^{p-1}\phi, \quad t \geqslant 0,\ x \in R^N, \tag{1}$$

where $\phi = \phi(t,x)$ is a complex-valued function of $(t,x) \in R^+ \times R^N$, Δ is the Laplace operator on R^N, $a > 0$; $N \geqslant 2$ and $1 < p < \frac{N+2}{(N-2)^+}$ (we use the convention: $\frac{N+2}{(N-2)^+} = \infty$ when $N = 2$, and $(N-2)^+ = N-2$ when $N \geqslant 3$, also see Wang [27]).

Concerning Eq. (1), there are nonlinear perturbations of the linear Klein–Gordon equation (1) with $a = 0$, for which there is a persistence of time-periodic and spatially localized solutions which are often called bound states. Meanwhile, (1) has the symmetry $\phi \mapsto \phi e^{i\gamma}$ and it has been shown [2] that (1) has time periodic and spatially localized solutions of the form $e^{i\omega t}\,u(x,\omega)$ with u in H^1, which bifurcate from the zero solution at the point eigenvalue of $-\Delta + V(x) - \omega^2$, by global variational [3,4], and local bifurcation methods [5].

Soffer and Weinstein in [1] considered a class of nonlinear Klein–Gordon equations with potentials:

$$\partial_t^2\phi = (\Delta - V(x) - m^2)\phi + af(\phi), \quad a \in R, \tag{2}$$

which are Hamiltonian and are perturbations of linear dispersive equations, and got some elegant results on the following three problems:

(P1) Do small amplitude spatially localized and time-periodic solutions persist for typical nonlinear and Hamiltonian perturbations?

(P2) What is the character of general small amplitude solutions to the perturbed dynamics?

(P3) How are the structures of the unperturbed dynamics manifested in the perturbed dynamics?

Moreover, they also obtained the local existence theory on the Cauchy problem for (2), which is the foundation of our paper. In the present paper, we do not consider the Eq. (2) with general nonlinearity, but consider the case $f(\phi) = |\phi|^{p-1}\phi$ in Eq. (2), namely, we consider Eq. (1) and are devoted to solving some problems which are different from (P1)–(P3) in [1] under appropriate assumptions on $V(x)$ by using other approaches.

We assume that $V(x)$ satisfies

$$V(x) \geqslant 0. \tag{3}$$

In the case of $V(x) < 0$, whether the results of the present paper hold or not is unknown in the literature and it will be our next work. Because there exists natural relation between the Schrödinger equation and the Klein–Gordon equation, let us recall the nonlinear Schrödinger equation with potentials