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A note on the reducibility of special infinite series

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Abstract

In this article we present how to reduce the evaluation of the special infinite series $S_n = \sum_{k=1}^{\infty} \frac{k^n}{(k+1)!}$, $n = 1, 2, 3, \ldots$ which are very time consuming in Computer Algebra Systems (CAS). An algorithm is given for this purpose. Based on the new algorithm, a fast and reliable MAPLE procedure is designed and $S_1, S_2, \ldots, S_{50}$ are given as sample output of this procedure together with the cpu time.

Keywords: Stirling numbers; Bell numbers; Algorithm; Series; Computer algebra systems

1. Introduction

The Stirling numbers of the second kind [1,2] describe the number of ways a set with n elements can be partitioned into k disjoint, non-empty subsets. These numbers usually denoted by $S(n,k)$. In [3] it is shown that by introducing the operator

$$\theta \equiv xD \equiv x\frac{\mathrm{d}}{\mathrm{d}x} \tag{1.1}$$

then we have

$$\theta^n \mathrm{e}^x = \sum_{k=1}^{\infty} \frac{k^n x^k}{k!} \tag{1.2}$$

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and

$$\theta^n e^x = e^x \sum_{k=1}^{n} S(n,k)x^k. \tag{1.3}$$

The motivation of the current paper is to use (1.2) and (1.3) in order to develop an algorithm to evaluate the infinite series of the form $S_n = \sum_{k=1}^{\infty} \frac{k^n}{(k+1)!}$, $n = 1, 2, 3, \ldots$ efficiently (for $n = 0$ it is known that $S_0 = e - 2$). The main results are presented in Section 2.

2. The main results

From (1.2) and (1.3) we obtain

$$\sum_{k=1}^{\infty} \frac{k^n x^k}{k!} = \sum_{k=1}^{n} S(n,k)x^k e^x. \tag{2.1}$$

Integrating both sides of (2.1) with respect to x between 0 and 1 (use integration by parts on the right hand side) yields

$$S_n = \sum_{k=1}^{\infty} \frac{k^n}{(k+1)!} = \sum_{k=1}^{n} S(n,k)I_k, \tag{2.2}$$

where $I_1 = 1$ and $I_k = e - kI_{k-1}$, $k = 2, 3, \ldots, n$. It turns out that

$$S_n = \sum_{k=1}^{\infty} \frac{k^n}{(k+1)!} = (-1)^{n+1}\left[1 - e\sum_{k=0}^{n}(-1)^k \binom{n}{k} B(k)\right], \quad n = 1, 2, 3, \ldots, \tag{2.3}$$

where $B(k)$ are the Bell numbers [2] which are related to the Stirling numbers of the second kind by

$$B(n) = \sum_{k=0}^{n} S(n,k), \quad n = 0, 1, 2, 3, \ldots \tag{2.4}$$

These numbers satisfy the recurrence relation [2]

$$B(0) = 1, \qquad B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k), \quad n = 0, 1, 2, \ldots \tag{2.5}$$

A significant outcome of the previous discussion is that the evaluation of the infinite series $S_n = \sum_{k=1}^{\infty} \frac{k^n}{(k+1)!}$, $n = 1, 2, 3, \ldots$ is now reduced to the evaluation of finite series either by using (2.2) or (2.3). This is very useful in computation particularly in Computer Algebra Systems (CAS) such as Maple, Mathematica, Macsyma and Matlab. Of course, the formula (2.3) should be preferred in the CAS computations since the computation of the Bell numbers are much faster than the computation of the Stirling numbers of the second kind.