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Dispersion induced by the pollution for the wave equation ☆

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Dedicated to my daughter Bhavika on the occasion of her first birthday

Abstract

In this paper we give a computational method to measure the 'dispersion' which is directly related to the 'pollution effect'. This pollution effect is observed by various researchers when, the Helmholtz equation, is solved for high wave numbers. The main effect of the 'pollution' is that the wave number of the solution obtained via a finite element method is different from the wave number of the exact solution and this is what is called 'dispersion'. Using Fourier analysis, we present a numerical method to measure the dispersion corresponding to a finite element method.

Keywords: Helmholtz equation; Pollution; Dispersion; Finite element method; Partial differential equation; Fourier analysis

1. Introduction

Helmholtz equation (with exterior radiation boundary condition) describes the electromagnetic scattering of time-harmonic waves. Various finite element techniques have been proposed in recent years to solve this differential equation. Out of these methods the standard Galerkin finite element method is widely used for such non-selfadjoint elliptic equations and it is observed by the various researchers that this method exhibit a variety of deficiencies, including

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oscillations or non-monotonicity of the solution and poor approximation of its derivatives. Substantial progress in overcoming the deficiencies of standard Galerkin method has been made subsequently.

It is known that asymptotically, for $h \to 0$, the Galerkin solution converges with the same rate as the best approximation. However, in preasymptotic range, the error of the Galerkin solution is significantly large than the error of the best approximation and this increases substantially with increasing k. (Here k denotes the wave number whereas h is the mesh width.) This behaviour of the FEM is conventionally termed as the *pollution effect* [4,24] of the method. The main effect of the 'pollution' is that the wave number of the solution obtained via a finite element method is different from the wave number of the exact solution and this is what is called 'dispersion'.

Researchers have observed that the standard Galerkin finite element method is polluted for one-dimensional problems but by some suitable modifications of the corresponding discrete bilinear form, this pollution can be eliminated. However, for two (and higher) dimensional problems no such modifications can give either the FEM with minimal pollution [2] or the pollution free FEM [4]. To reduce the pollution error, several numerical methods have been proposed. These are, the Galerkin least squares method [21,32], the generalized finite element method [2], the partition of unity method [3,26], subgrid finite element method [28], element free Galerkin method [31]. Some other articles which deal with the reduction of pollution effect are [16–18,20,26].

In this paper, we present a numerical method to measure the dispersion corresponding to any finite element method. Our FOSLS methodology involves the introduction of a new variable. This has the effect of transforming the second-order elliptic problem into a system of first-order. By means of numerical experiments, we will show that the regularized FOSLS method have no dispersion whereas the standard Galerkin has. Such FOSLS formulations have been considered earlier by several researchers. For example, one can refer to [5–8,10–14,29], and the references therein.

The paper is organized as follows. In Section 2, we describe the main problem. Two types of finite element discretizations are described in Section 3. Since the pollution effect is related to the phase lag $k - k_h$, in Section 4 we will study this phase lag via dispersion analysis. Finally, the numerical results for this dispersion analysis along with the discussion are given in Section 5.

2. Problem description

Let $D \in \mathscr{R}^2$ be a bounded domain, which will be assumed to have a $C^{1,1}$ (i.e., first derivative is Lipschitz continuous) or convex polygonal boundary of positive measure. Consider the exterior Helmholtz boundary value problem