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# Existence results for $n$-point boundary value problem of second order ordinary differential equations ☆

Shihua Chen[a,*], Jia Hu[a], Li Chen[b], Changping Wang[c]

[a]College of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China
[b]The State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, PR China
[c]Department of Mathematics and Statistics, Dalhousie University, Halifax NS, Canada B3H 3J5

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## Abstract

This study concerns the existence of positive solutions to the boundary value problem

$$u'' + a(t)f(u) = 0, \quad t \in (0, 1),$$

$$u'(0) = \sum_{i=1}^{n-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n-2} a_i u(\xi_i),$$

where $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, $a_i, b_i \in [0, \infty)$ with $0 < \sum_{i=1}^{n-2} a_i < 1$ and $\sum_{i=1}^{n-2} b_i < 1$. By applying the Krasnoselskii's fixed-point theorem in Banach spaces, some sufficient conditions guaranteeing the existence of at least one positive solution or at least two positive solutions are established for the above general $n$-point boundary value problem.


*Keywords:* Positive solution; Fixed point theorem; $n$-point boundary value problem; Sufficient conditions

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* Corresponding author.
*E-mail address:* shcheng@s1000e.wuhee.edu.cn (S. Chen).

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## 1. Introduction

The multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [6]. Since then, nonlinear multi-point boundary value problems have been studied by several authors using the Leray-Schauder Continuation theorem, coincidence degree theory, and fixed-point theorem in cones [1,2,5]. We refer the reader to [10,3,9,4,8] for other recent results on nonlinear multi-point boundary value problems.

Recently, Ma and Castaneda [10] studied the existence problem of the general $n$-point boundary value problem

$$u'' + a(t)f(u) = 0, \quad t \in (0,1),$$

$$u'(0) = \sum_{i=1}^{n-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n-2} a_i u(\xi_i), \tag{1.1}$$

under the following assumption:

(A1) $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$. $a_i, b_i \in [0,\infty)$ $(i = 1, 2, \ldots, n-2)$ satisfying $0 < \sum_{i=1}^{n-2} a_i < 1$ and $\sum_{i=1}^{n-2} b_i < 1$.

(A2) $f \in C([0,\infty),[0,\infty))$; $a \in C([0,1],[0,\infty))$ and $a(t) \not\equiv 0$ on $[0,1]$.

They show that the boundary value problem (1.1) has at least one positive solution in one of the following cases

(i) $f_0 = 0$ and $f_\infty = +\infty$ (Superlinear case),
(ii) $f_0 = +\infty$ and $f_\infty = 0$ (Sublinear case),

where

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}. \tag{1.2}$$

It is natural to put forward problems such as, whether or not we can obtain a similar conclusion if $f_0 = f_\infty = 0$ or $f_0 = f_\infty = \infty$ and whether or not we can obtain a similar conclusion if $f_0, f_\infty \notin \{0, \infty\}$.

The purpose of this paper is to generalize the results in [10] and to establish some sufficient conditions for the existence of positive solutions to the boundary value problem (1.1) without the superlinear condition or sublinear condition, which gives a positive answer to the questions stated above. The key tool in finding our main results is the following well-known fixed-point theorem due to Krasnoselskii [7].

**Theorem 1.1.** *Let $E$ be a Banach space and $K$ be a cone in $E$. Assume $\Omega_1$ and $\Omega_2$ are open subsets of $E$, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ be a completely continuous operator such that*