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Fisher information of orthogonal hypergeometric polynomials

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Abstract

The probability densities of position and momentum of many quantum systems have the form $\rho(x) \propto p_n^2(x)\omega(x)$, where $\{p_n(x)\}$ denotes a sequence of hypergeometric-type polynomials orthogonal with respect to the weight function $\omega(x)$. Here we derive the explicit expression of the Fisher information $I = \int \mathrm{d}x[\rho'(x)]^2/\rho(x)$ corresponding to this kind of distributions, in terms of the coefficients of the second-order differential equation satisfied by the polynomials $p_n(x)$. We work out in detail the particular cases of the classical Hermite, Laguerre and Jacobi polynomials, for which we find the value of Fisher information in closed analytical form and study its asymptotic behaviour in the large n limit.

Keywords: Classical orthogonal polynomials; Fisher information; Second-order differential equations; Probability measures

1. Introduction

Let X denote a continuous random variable with probability density function $\rho(x)$ (for the sake of simplicity, we confine ourselves to the one-dimensional case). The Fisher information corresponding to

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this probability distribution is defined as

$$I(X) = \int \frac{[\rho'(x)]^2}{\rho(x)}\,\mathrm{d}x. \tag{1}$$

This quantity was first introduced in the framework of statistical estimation theory, where it plays a key role [4]. It is related to the standard deviation of X by means of a particular case of the so-called Cramér–Rao inequality (see e.g. [1, Chapter 12]),

$$(\Delta X)^2 I(X) \geqslant 1, \tag{2}$$

and is also related in a similar way to the Boltzmann–Shannon information entropy [1,18],

$$H(X) = -\int \rho(x) \log\, \rho(x)\,\mathrm{d}x. \tag{3}$$

Specifically, defining the entropy power of X as

$$N(X) = \frac{1}{2\pi e}\, \exp[2H(X)], \tag{4}$$

it holds the uncertainty inequality [19]

$$I(X)N(X) \geqslant 1. \tag{5}$$

Let us remark that Shannon entropy and Fisher information are two complementary measures of spreading of the density ρ, the former being of global character because of the logarithm and the latter having a local character due to the gradient [1].

It can be shown [1,19] that, in both (2) and (5), equality is attained if and only if the probability density is a Gaussian,

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}}\, \exp\left(-\frac{(x-x_o)^2}{2\sigma^2}\right),$$

for which we have

$$I(X) = \frac{1}{\sigma^2}, \quad (\Delta X)^2 = N(X) = \sigma^2.$$

Fisher information can also be evaluated in closed analytical form for many other probability distributions. For instance, for the Student-t distributions

$$f(m,\sigma,x) = \frac{\Gamma((m+1)/2)}{\sigma\sqrt{m-2}\,\Gamma(m/2)\Gamma(1/2)}\left(1+\frac{x^2}{(m-2)\sigma^2}\right)^{-(m+1)/2}, \quad m>2,$$

the Fisher information has the expression [20]

$$I(m,\sigma) = \frac{m(m+1)}{(m-2)(m+3)\sigma^2},$$