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Optimal order error estimates for finite element approximations of a bifurcation function

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Abstract

The bifurcation function associated with an elliptic boundary value problem $Au + g[u] = 0$ is a vector field $B(\omega)$ on $\mathbb{R}^d$ with the property that the solutions of the boundary value problem are in a one-to-one correspondence with the zeros of B. A finite element approximation B^h of B is formulated and optimal order error estimates are derived. Implementation issues are also discussed.

MSC: 65N15; 65N30

Keywords: Bifurcation function; Finite element approximation; Semilinear elliptic boundary value problem

1. Introduction

The problem of determining the set of solutions $\mathscr{S}(\lambda)$ of a semilinear elliptic boundary value problem, $Lu + g[u, \lambda] = 0$, has been considered by many authors. In general, this is a problem of global analysis,

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although many of the methods used are local in natural. For example, continuation methods and the method of Lyapunov–Schmidt. Continuation methods (see [13] for a general development) assume a known noncritical solution (u_0, λ_0), and obtain nearby solutions by varying λ in a neighborhood of λ_0. The implicit function theorem guarantees that there is a unique curve of solutions to be found. In the method of Lyapunov–Schmidt (cf. [9]), the implicit function theorem is used to reduced the problem to the null space of the linearization at a critical point (u_0, λ_0). It thus reduces the problem, in the neighborhood of the critical point, to a finite-dimensional problem.

For an important class of problems, there is another approach that applies directly to the global problem. Using the method of alternative problems (cf. [8,9]) the elliptic problem can be shown to be equivalent to a finite dimensional problem. In this way the global problem is recast as a vector field equation $B(\omega, \lambda) = 0$, $\omega \in \mathbb{R}^d$, for some (typically small) integer d. The vector field, $B(\omega, \lambda)$ is called the bifurcation function. The reduction process that leads to the bifurcation equation, $B(\omega, \lambda) = 0$, can be regarded as a technique for correcting Galerkin's method so that it becomes exact. However, by construction, the bifurcation function depends on solutions of a nonlocal boundary value problem, and hence this information is not readily accessible analytically except in some special cases. To extend the utility of the bifurcation function numerical methods are needed.

Theoretically, the process relies on the use of spectral quantities; namely the eigenfunctions of a linear operator. As a consequence, when the eigenfunctions are known they can be used to develop a spectral numerical method that mimics the actual construction. This approach relies on special operators and geometries. To allow for more general situations we presented a finite element method in [14] for computing the bifurcation function for self-adjoint problems. In this work we show that the method extends to elliptic problems that are not self-adjoint, and at the same time verify optimal order error estimates. In our previous work, we were only able to verify sub-optimal error estimates. The proofs presented here are also considerably simplier than those given in our previous work.

Several other authors have contributed to the numerical analysis of continuation and bifurcation methods for elliptic boundary value problems. Among the first were Kikuchi [11,12] and Brezzi et al. [5–7]. Approximating a branch of solutions without bifurcation points was discussed in the works [5,6,12], while the case of approximating bifurcating curves of solutions, in the neighborhood of a simple bifurcation point, was considered in [7,11]. A numerical version of the method of Lyapunov–Schmidt was proposed by Bohmer and Mei in [2,1] for approximating branching manifolds in the neighborhood of a possibly higher order bifurcation point. See the papers [3,4] for extensions of these earlier works. In contrast to these works, our work focuses on characterizing and computing the entire set of solutions $\mathscr{S}(\lambda)$, for a given value of λ that is typically not a bifurcation point.

In Section 2 we outline the reduction process that leads to the bifurcation function for a semilinear elliptic boundary value problem. A finite element method for computing approximations to the bifurcation function is also described there. In Section 3 proofs of optimal error estimates are given. Necessarily, error estimates of the solutions of the associated nonlocal boundary value problem are also established. Estimates on approximate solutions of the boundary value problem also follow from these estimates since all solutions must lie on a manifold determined by the nonlocal problem. Section 4 discusses the resulting algebraic problems that arise at the level of matrix–vector representations. Computational examples of the use of the bifurcation function to determine all solutions of a boundary value problem and their stability properties can be found in [15].