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On a system of nonlinear PDE's for phase transitions with vector order parameter and convective effect

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Abstract

The paper deals with a system of nonlinear PDE's which describes a phase-field model with convection and temperature dependent constraint to the vector order parameter. Existence of solutions for the system under consideration is proved by the method of Yosida approximation and fixed point arguments.

MSC: 35R70; 35K50; 74N30

Keywords: Nonlinear PDE's; Existence of solutions; Subdifferential; Yosida approximation; Hysteresis; Vector order parameter; Convection

1. Introduction

The present paper deals with a class of phase transition models which takes into account the hysteresis, diffusive and convective effects and is described by the following system of PDE's

$$a\mathbf{w}_t + \partial \mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u) \quad \text{in } Q, \tag{1}$$

$$(c_1 w_1 + c_2 w_2)_t + du_t - \nabla \cdot (\nabla u + \hat{\mathbf{K}}(\mathbf{w})) = h(\mathbf{w}, u) \quad \text{in } Q, \tag{2}$$

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where $\mathbf{w} = (w_1, w_2)$, $T > 0$, $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $Q = (0, T) \times \Omega$; $a,\ c_1,\ c_2,\ d$ are given constants; $\hat{\mathbf{K}} : R^2 \to R^N$, $F : R^2 \times R \to R^2$, $h : R^2 \times R \to R$, $f_*, f^*, g_*, g^* : R \to R$ are given functions. We assume that $0 \leqslant b \leqslant 1$ is a given constant, $f_*, f^*, g_*, g^* \in C^2(R)$, $f_*(u) \leqslant f^*(u)$ on R, $g_*(u) \leqslant g^*(u)$ on R, $f_*(u),\ f^*(u),\ g_*(u),\ g^*(u)$ are nondecreasing functions on R. Also we suppose that there exists a constant $k_0 > 0$ such that $f_*(u) = f^*(u) = g_*(u) = g^*(u) = -1$ on $(-\infty, -k_0]$ and $f_*(u) = f^*(u) = g_*(u) = g^*(u) = 1$ on $[k_0, \infty)$.

Define

$$K(u) = \{(w_1, w_2) : \ f_*(u) \leqslant w_1 + bw_2 \leqslant f^*(u),\ \ g_*(u) \leqslant -bw_1 + w_2 \leqslant g^*(u)\}.$$

We denote by $\mathbf{I}_{K(u)}(\cdot)$ the indicator function of the set $K(u)$ and $\partial\mathbf{I}_{K(u)}(\cdot)$ denotes the subdifferential of $\mathbf{I}_{K(u)}(\cdot)$. The subdifferential $\partial\mathbf{I}_{K(u)}(\mathbf{w})$ is a set-valued mapping and in our statement of the problem $\partial\mathbf{I}_{K(u)}(\mathbf{w}) = \{0\}$ if $\mathbf{w} \in \operatorname{int} K$, and $\partial\mathbf{I}_{K(u)}(\mathbf{w})$ coincides with the cone of normals to K at the point $\mathbf{w}$ if $\mathbf{w} \in \partial K$.

In this paper we study system (1), (2) together with the following boundary and initial conditions:

$$\nu \cdot (\nabla u + \hat{\mathbf{K}}(\mathbf{w})) = 0 \quad \text{on } \Sigma = (0, T) \times \partial\Omega, \tag{3}$$

$$\mathbf{w}(0, x) = \mathbf{w}_0(x), \quad u(0, x) = u_0(x) \quad \text{in } \Omega, \tag{4}$$

where ν is the unit outward normal vector on $\partial\Omega$, $\mathbf{w}_0,\ u_0$ are given initial data.

Eqs. (1) and (2) correspond, respectively, to the kinetics of the vector order parameter $\mathbf{w} = (w_1, w_2)$ and the balance of the internal energy; u is the relative temperature. System (1), (2) describes solid–liquid phase transitions of a physical system which is a mixture of two substances having different solidification temperatures. If we consider melting problem of only one substance then the respective system contains scalar function for the order parameter. However, in the case when we have mixture of two different substances, the adequate mathematical model describing their solid–liquid phase transitions contains two component vector order parameter. Let us note that models with vector hysteresis are object of active recent investigations (see papers [7,21] as well as monograph [17]).

Various special cases of system (1), (2) have been already studied. In [19,20] Visintin proposed the following system:

$$aw_t + \partial I_{[-1,1]}(w) \ni a_1 w + a_2 u \quad \text{in } Q,$$

$$cw_t + du_t - \Delta u = g(x, t) \quad \text{in } Q$$

as a model for Stefan problem with phase relaxation, where $f_*(u) \equiv -1$, $f^*(u) \equiv 1$. Further investigations deal with the case of cubic nonlinearity $-a_0 w^3 + a_1 w + a_2 u$ in the kinetics of the order parameter, see [6,9,10]. The equation

$$w_t + \partial I_u(w) \ni 0 \quad \text{in } Q,$$

including the constraint $f_*(u) \leqslant w \leqslant f^*(u)$, was investigated in [3,19], see also [8,11,12]. In [5], Colli et al. studied system (1), (2) in the special case when w is scalar function, $h = h(t, x)$ and there is no convective effects: $\hat{\mathbf{K}} = 0$. Later, Kubo in [13] studied the case of scalar order parameter in the presence of convective effects.

Let us note that in mathematical aspect the present paper has been influenced by paper [13] as well as [5,18]. Our goal is to incorporate the case of vector order parameter to the phase transition phenomena with diffusive, hysteresis and convective effect.