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A symbolic operator approach to several summation formulas for power series

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Abstract

This paper deals with the summation problem of power series of the form $S_a^b(f;x)=\sum_{a\leqslant k\leqslant b} f(k)x^k$, where $0\leqslant a<b\leqslant\infty$, and $\{f(k)\}$ is a given sequence of numbers with $k\in[a,b)$ or $f(t)$ is a differentiable function defined on $[a,b)$. We present a symbolic summation operator with its various expansions, and construct several summation formulas with estimable remainders for $S_a^b(f;x)$, by the aid of some classical interpolation series due to Newton, Gauss and Everett, respectively.

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1. Introduction

It is known that the symbolic operations Δ (difference), E (displacement) and D (derivative) play an important role in the calculus of finite differences as well as in certain topics of computational methods.

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For various classical results, see, e.g., [7,8], etc. Certainly, the theoretical basis of the symbolic methods could be found within the theory of formal power series, in as much as all the symbolic expressions treated are expressible as power series in Δ, E or D, and all the operations employed are just the same as those applied to formal power series. For some easily accessible references on formal series, we may recommend [2,3,11].

Recall that the operators Δ, E and D may be defined via the following relations:

$$\Delta f(t) = f(t+1) - f(t), \quad Ef(t) = f(t+1), \quad Df(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t).$$

Using the number 1 as an identity operator, viz. $1f(t) = f(t)$, one can observe that these operators satisfy the formal relations

$$E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1+\Delta).$$

Powers of these operators are defined in the usual way. In particular, one may define for any real number x, $E^x f(t) = f(t+x)$.

Note that $E^k f(0) = \left[E^k f(t)\right]_{t=0} = f(k)$, so that any power series of the form $\sum_{k=0}^{\infty} f(k)x^k$ could be written symbolically as

$$\sum_{k\geqslant 0} f(k)x^k = \sum_{k\geqslant 0} x^k E^k f(0) = \sum_{k\geqslant 0} (xE)^k f(0) = (1-xE)^{-1} f(0).$$

This shows that the symbolic operator $(1-xE)^{-1}$ with parameter x can be applied to $f(t)$ (at $t=0$) to yield a power series or a generating function for $\{f(k)\}$.

We shall show in Section 3 that $(1-xE)^{-1}$ could be expanded into series in various ways to derive various symbolic operational formulas as well as summation formulas for $\sum_{k\geqslant 0} f(k)x^k$. Note that the closed form representation of series has been studied extensively. See, for example, [9] which presents a unified treatment of summation of series using function theoretic method. Some consequences of the summation formulas as well as the examples will be shown in Section 4, can be useful for computational purpose, accelerating the series convergence. In Section 5, we shall give the remainders of the summation formulas.

2. Preliminaries

We shall need several definitions as follows.

Definition 2.1. The expression $f(t) \in C^m_{[a,b)}$ $(m \geqslant 1)$ means that $f(t)$ is a real function continuous together with its mth derivative on $[a,b)$.

Definition 2.2. $\langle x, x_0, x_1, \ldots, x_n \rangle$ represents a least interval containing x and the numbers $x_0, x_2, \ldots, x_n$.

Definition 2.3. $\alpha_k(x)$ is called an Eulerian fraction and may be expressed in the form (cf. [3])

$$\alpha_k(x) = \frac{A_k(x)}{(1-x)^{k+1}}, \quad (x \neq 1),$$