Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 176 (2005) 411–424

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Inverse spherical surfaces

G. Casciola[1], S. Morigi[*,1]

Department of Mathematics, University of Bologna, P.zza di Porta San Donato 5, 40127 Bologna, Italy

Received 9 January 2004

Abstract

In this paper we describe a new way to design rational parametric surfaces defined on spherical triangles which are useful for modelling in a spherical environment. These surfaces can be seen as single-valued functions in spherical coordinates.

Keywords: Geometric modelling; Spherical coordinates; Rational triangular Bézier patch; Single-valued surfaces

1. Introduction

The problem of defining curves on curves, or surfaces on surfaces, plays an important role in computer aided geometric design (CAGD), in particular, the problem of defining curves and surfaces over the sphere is of a certain interest since it allows us to model circular/spherical phenomena in a more natural way. The reader is referred to [8, Chapter 9], for a detailed description of this subject.

In this work we address the problem of defining convenient modelling tools involving patches defined over spherical triangles. This problem has not received much attention in the literature, possibly because until recently it was incorrectly believed that there were no suitable forms of spherical barycentric coordinates. This myth was dispelled in [2] where coordinates used already more than 100 years ago by Möbius were employed to create the so-called CBB curves on circular arcs [1], and their generalization, called SBB-patches, on spherical triangles [2]. These patches turn out to be suitable for data fitting on the sphere [3], although, as already observed in [2], they are not particularly useful for design purposes because, in

* Corresponding author. Tel.: +390512094449; fax: +390512094490.
E-mail addresses: casciola@dm.unibo.it (G. Casciola), morigi@dm.unibo.it (S. Morigi).
[1] This research was supported byMIUR-Cofin2003 project, and by University of Bologna “Funds for selected research topics”.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.07.030

general, they are not close to their control curve/surface. To overcome this deficiency, interesting proposals for modelling curves on circular arc were presented in [14,15], and known as polar(p)-Bézier curves and in [4,16], and named polar(p)-spline curves, while their generalization to tensor product surfaces is given in [17]. Unfortunately, an extension of p-Bézier curves to spherical patches on spherical triangles, due to the geometry of the sphere, does not exist.

In fact, the key aspect of the curve proposal in [15] is a fan-transformation of the circular arc of a factor n that gives an n-partition on the arc defined by n sub-arcs of the same arc length. This concept has no natural generalization to the spherical setting. Let us motivate this issue in more details.

An n-partition of a planar triangle $\mathscr{T} := \langle u_1, u_2, u_3\rangle$, consists of n^2 identical sized and shaped triangles on $\mathscr{T}$. It is well-known that all the triangles of this n-partition on $\mathscr{T}$ have edges of lengths $\langle u_1, u_2\rangle$, $\langle u_2, u_3\rangle$, $\langle u_1, u_3\rangle$ divided by a factor n.

As observed in [2], for general $n > 1$, there is no analogous way to partition a spherical triangle $T := \langle v_1, v_2, v_3\rangle$, with geodesic boundaries. That is, the sub-triangles of the n-partition of T have boundaries that are not given by a reduction of a factor n of the geodesic lengths of the boundaries of T: $\langle v_1, v_2\rangle$, $\langle v_2, v_3\rangle$, $\langle v_1, v_3\rangle$.

For a better understanding, let us see the case $n = 2$. Connecting, for example, the middle points of $\langle v_1, v_2\rangle$ and $\langle v_1, v_3\rangle$ of a spherical triangle T we get an arc of length x on the great circle through these points. From spherical trigonometry, $\cos(x) = \cos(A)\cos(\langle v_2, v_3\rangle/2)$ where A is the area of the spherical triangle T. This trivially shows that $x \neq \langle v_2, v_3\rangle/2$.

In this paper we characterize a special subset of rational Bézier patches defined on spherical triangular domains that allows us to model with single-valued surfaces in spherical coordinate system. These patches, that we call inverse spherical surfaces (ISS), exhibit most of the important modelling properties such as for example a good sketch of their control surface. Moreover, they also offer a natural way to model surfaces with complex shapes on the sphere, while modelling such surfaces with rectangular patches would require degenerate patches.

The ISS can be proposed as basic representation for a spherical modelling environment which can be seen as a powerful tool in a classical CAD system based on nonuniform rational B-splines (NURBS) in order to extend its potentiality.

The paper is organized as follows. In Section 2 we briefly review some notations and basic facts about rational Bézier patches defined on triangular domains. In Section 3 we consider a special class of rational Bézier patches as ISS, and in Section 4 surfaces in this class are analyzed in the spherical setting as single-valued surfaces. Our proposal of a simple ISS subclass, efficient and useful for modelling, is described in Section 5, and in Section 6 we discuss how these ISSs can be smoothly joined together. We conclude the paper with remarks in Section 7.

2. Rational triangular Bézier patches

In this section we recall some well-known facts about rational triangular Bézier patches, see [6,10]. Let T^* be a triangle with vertices in $\mathbb{R}^2$. Given a point $\xi \in T^*$, let $(\alpha_1, \alpha_2, \alpha_3)$ be its barycentric coordinates relative to T^*. Then the Bernstein basis polynomials of degree n on T^* are defined by

$$B^n_{ijk}(\xi) := \frac{n!}{i!\,j!\,k!}\,\alpha_1^i \alpha_2^j \alpha_3^k, \tag{1}$$