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A steepest descent method for vector optimization

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Abstract

In this work we propose a Cauchy-like method for solving smooth unconstrained vector optimization problems. When the partial order under consideration is the one induced by the nonnegative orthant, we regain the steepest descent method for multicriteria optimization recently proposed by Fliege and Svaiter. We prove that every accumulation point of the generated sequence satisfies a certain first-order necessary condition for optimality, which extends to the vector case the well known “gradient equal zero” condition for real-valued minimization. Finally, under some reasonable additional hypotheses, we prove (global) convergence to a weak unconstrained minimizer.

As a by-product, we show that the problem of finding a weak constrained minimizer can be viewed as a particular case of the so-called Abstract Equilibrium problem.

Keywords: Pareto optimality; Vector optimization; Steepest descent; K-convexity; Quasi-Féjer convergence

1. Introduction

In multicriteria optimization, several objective functions have to be minimized simultaneously. Usually, no single point will minimize all given objective functions at once (i.e., there does not exist an *ideal* minimizer), and so the concept of optimality has to be replaced by the concept of *Pareto-optimality* or *efficiency*. A point is called Pareto-optimal or efficient, if there does not exist a different point with

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smaller than or equal objective function values, such that there is a decrease in at least one objective function value. Applications for this type of problem can be found in engineering design [12] (mainly truss optimization [8]), location science [5], statistics [6], management science [13] (specially portfolio analysis [24]), etc.

Among the main solution strategies for multicriteria optimization problems, we mention the scalarization approaches [15,16,19,21,23]. Here, one or several parameterized single-objective (i.e., classical) optimization problems are solved. Frequently, some parameters have to be specified in advance, leaving the modeler and the decision-maker with the burden of choosing them. Moreover, in the weighting method, for example, bad choices of these parameters can lead to unbounded scalar problems. Other scalarization techniques are parameter-free [7,8,21] but try to compute a discrete approximation to the whole set of Pareto-optimal points.

Parameter-free multicriteria optimization techniques use in general an ordering of the different criteria, i.e., an ordering of importance of the components of the objective function vector. In this case, the ordering has to be specified. Moreover, the optimization process is usually augmented by an interactive procedure [20], adding an additional burden to the task of the decision-maker.

In a recent paper, Fliege and Svaiter [14] proposed a Pareto descent method for multiobjective optimization. This procedure is parameter-free and relies upon a suitable extension for vector-valued functions of the classical steepest descent direction. Neither ordering information nor weighting factors for the different objective functions is assumed to be known in this new method, which may be interpreted as a "Cauchy's method" for multicriteria optimization.

We recall that the steepest descent method (also known as gradient or Cauchy's method) is one of the oldest and more basic minimization schemes for scalar unconstrained optimization. Despite its computational shortcomings, like, for instance, "hemstitching" phenomena, the Cauchy's method can be considered among the most important procedures for minimization of real-valued functions defined on $\mathbb{R}^n$, since it is the departure point for many other more sophisticated and efficient algorithms. For instance, it is partially used in some "globally convergent" modifications of Newton's method for unconstrained optimization. Here, "globally convergent" means that all sequences produced by the method have decreasing objective function values, and that all accumulation points of these sequences are critical points. We refer the reader to [9], where the "double dog-leg" method is discussed. The simple idea of decreasing the value of the objective function is also used in many other modifications of Newton's method. We refer the reader again to [9] for a very clear exposition. It remains an open question how to extend more efficient procedures, as Newton's method, to vector optimization.

The purpose of this paper is to take a step further on the direction of Fliege and Svaiter's [14] work. Based on their ideas, we present a Cauchy-like method for smooth vector optimization. In this setting, the partial order is induced by a general closed convex pointed cone K, with nonempty interior in a finite-dimensional space. Our procedure depends on the choice of an arbitrary initial point, as well as on a certain compact set which characterizes the positive polar cone (see Section 3). On the final remarks we make more comments on this issue. When the cone is the nonnegative orthant, our procedure turns out to be the very same proposed by them for the unconstrained case. Our convergence results extend theirs and, furthermore, under some additional (and quite reasonable) hypotheses, we also show that all sequences produced by the method converge to a weakly efficient point, no matter how poor is the initial guess. We point out that we are not attempting to find the set of all efficient or weak efficient optima.

Regarding the importance of vector optimization, we point out that even though the vast majority of real life problems formulated as vector-valued problems deals with the component-wise partial order, i.e.,