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Almost sure asymptotic stability of drift-implicit θ-methods for bilinear ordinary stochastic differential equations in $\mathbb{R}^1$

Alexandra Rodkina[a], Henri Schurz[a,b,c,*]

[a] *Department of Mathematics and Computer Science, University of the West Indies at Mona, Kingston, 7, Jamaica*
[b] *Department of Mathematics, Southern Illinois University, 1245 Lincoln Drive, Carbondale, IL 62901-4408, USA*
[c] *Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA*

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Abstract

Global almost sure asymptotic stability of stochastic θ-methods with nonrandom variable step sizes when applied to bilinear, nonautonomous, homogeneous test systems of ordinary stochastic differential equations (SDEs) is investigated. Sufficient conditions for almost sure asymptotic stability are proved for both analytical and numerical solutions in $\mathbb{R}^1$. The results of Saito and Mitsui (World Sci. Ser. Appl. Math. 2 (1993) 333, SIAM J. Numer. Anal. 33 (1996) 2254), Higham (SIAM J. Numer. Anal. 38 (2001) 753) and Schurz (Stochastic Anal. Appl. 14 (1996) 313, Handbook of Stochastic Analysis and Applications, 2002) for the constant step sizes are carried over to the case with variable step sizes and nonautonomous linear test equations. The investigations indicate that θ-methods with variable step sizes or variable parameter θ governed by certain conditions can successfully be used to guarantee almost sure asymptotic stability while discretizing nonautonomous SDEs.

Keywords: Stochastic differential equations; Nonautonomous test equations; Numerical methods; Variable step sizes; Global asymptotic stability; Almost sure stability; Drift-implicit θ-methods

* Corresponding author. Department of Mathematics, Southern Illinois University, 1245 Lincoln Drive, Carbondale, IL 62901-4408, USA. Tel.: +618 4536580; fax: +618 4535300.

E-mail address: hschurz@math.siu.edu (H. Schurz).

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1. Introduction

Several authors have already investigated the stability behavior of stochastic versions of θ-methods (or their subclasses)

$$Y_{n+1} = Y_n + \theta_n \alpha_{n+1} Y_{n+1} \Delta_n + (1 - \theta_n)\alpha_n Y_n \Delta_n + \sigma_n Y_n \Delta W_n, \tag{1}$$

where $\alpha_n, \sigma_n, \theta_n \in \mathbb{R}^1$ are nonrandom parameters, $\Delta_n = t_{n+1} - t_n$ is the current step size along discretizations $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n_T} = T$ of intervals $[0, T]$, driven by martingale-differences ΔW_n with $\mathbb{E}[(\Delta W_n)^2] = \Delta_n < +\infty$ (Typical choices for ΔW_n are Gaussian distributed $\Delta W_n \in \mathcal{N}(0, \Delta_n)$ for strong approximations of SDEs or two-point distributed ΔW_n with probabilities $\mathbb{P}\left(\Delta W_n = \pm\sqrt{\Delta_n}\right) = \frac{1}{2}$ for weak approximations of SDEs.) For example, see Artemiev [2,3], Artemiev and Averina [4], Burrage et al. [5], Higham [7,8], Kloeden and Platen [13], Kloeden et al. [14], Ryashko and Schurz [22], Saito and Mitsui [23], Schurz [25,26,28–31], or Talay [35]. Most of the mentioned works only deal with asymptotic stability of moments with respect to autonomous testequations of one-dimensional SDEs and are restricted to numerical algorithms with constant step sizes. Higham [8], Saito and Mitsui [23], Schurz [30,31] have already discussed almost sure stability of stochastic θ-methods with constant step sizes and for autonomous test equations. Nonlinear test equations and almost sure asymptotic stability of linear-implicit generalizations of θ-methods are studied in Rodkina and Schurz [20,21]. It is also worth noting that the class of numerical methods (1) contains the widely used forward Euler–Maruyama methods by choosing $\theta_n = 0$ for all n, the backward Euler–Maruyama methods by taking $\theta_n = 1$ for all n and the trapezoidal methods (here, in the linear case, identical with the midpoint methods) by setting $\theta_n = 0.5$ for all $n \in \mathbb{N}$. Moreover, from moment analysis with equidistant step sizes, we know that $\theta_n = 0.5$ (i.e. the symmetric cases of trapezoidal or midpoint methods) is a preferrable choice for "adequate" stochastic-numerical integration, cf. Schurz [27]. Whether this is the case for the requirement of almost sure asymptotic stability too is still an open question. Besides, the problem of appropriate test equations has not been solved in stochastics so far, i.e. the relevance of test equations for which class of SDEs is not clarified. This fact alone urges us to analyze numerical methods for nonautonomous or nonlinear stochastic test equations.

In the present paper we investigate almost sure asymptotic stability of methods (1) with variable, nonrandom step sizes and with respect to nonautonomous test equations of linear SDEs. In particular, we shall extend results of Saito and Mitsui [23], Higham [8] and Schurz [31] which are only known for constant step sizes and linear, autonomous test equations so far. For this purpose, throughout this paper, we assume that their driving noise terms ΔW_n are independent random variables with finite first moments $\mathbb{E}[\Delta W_n] = 0$ and second moments $\mathbb{E}[(\Delta W_n)^2] = \Delta_n$, and we interpret these methods as numerical approximations of bilinear, nonautonomous test equations of Itô SDEs

$$\mathrm{d}X(t) = \alpha(t)X(t)\,\mathrm{d}t + \sigma(t)X(t)\,\mathrm{d}W(t) \tag{2}$$

driven by standard Wiener process $W = \{W(t) : t \geqslant 0\}$. New criteria for the a.s. asymptotic stability of the zero solution of SDEs (2) are stated and proven too. For the sake of technical simplicity, we suppose that the presented analysis is done on the base of a completed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ with natural filtration $\mathcal{F}_t = \sigma(W(s) : 0 \leqslant s \leqslant t)$, and the initial value $X(0) = x_0$ is known and independent of $\mathcal{F}_{+\infty}$. Eventually we are going to compare the asymptotic behavior of the analytic solution of (2) to its numerical approximations (1) with $\alpha_n = \alpha(t_n)$ and $\sigma_n = \sigma(t_n)$, driven by $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$-martingale-differences ΔW_n (for details on martingales, see [15]) with $\mathcal{F}_n = \mathcal{F}_{t_n}$. Let $\mathcal{B}(S)$ denote the set of all Borel-sets of inscribed set S and μ the Lebesgue measure in $\mathbb{R}^1$.